Notation and preliminaries

Notation for path diagrams

$\boxed{x_1}$ observed variables

ξ_1 (in ellipse) latent variables

δ_1 error terms (unenclosed)

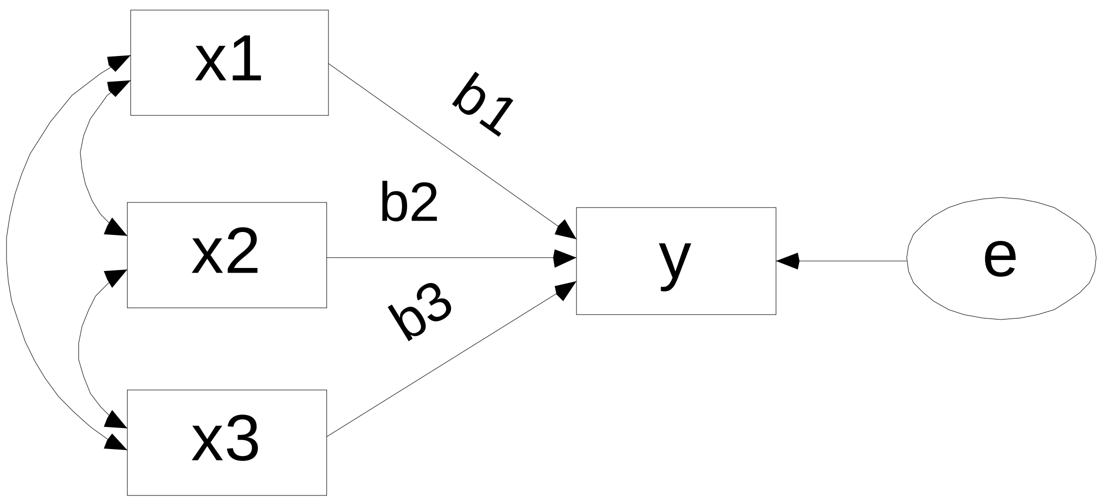
$\longrightarrow$ causal path or factor loading/weight

covariance or unconstrained (nonzero) relationship between exogenous variables

Multiple regression

$$y = b_1 x_1 + b_2 x_2 + b_3 x_3 + e$$

$$= \mathbf{b}'\mathbf{x} + e$$

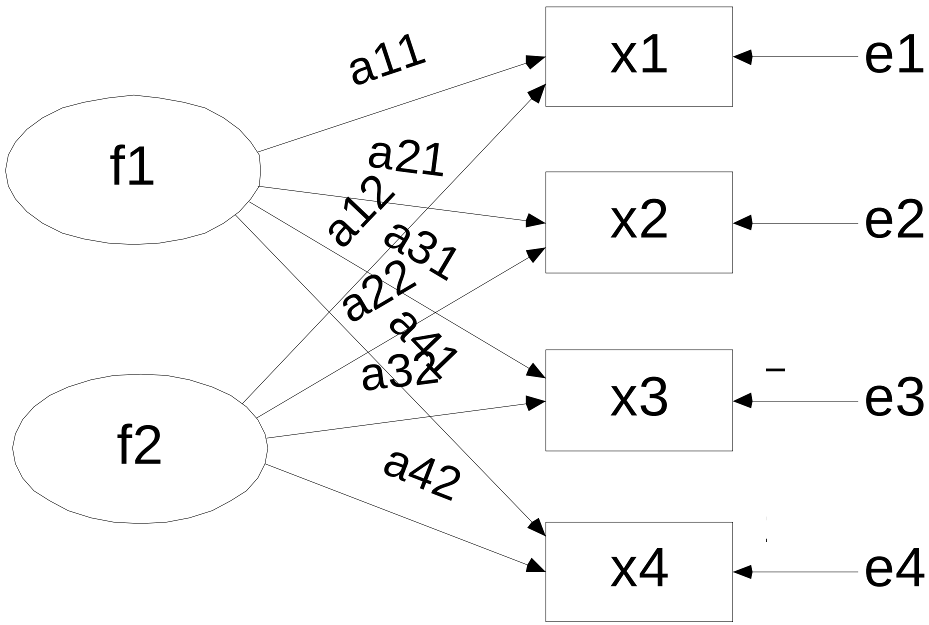


- How could the intercept term enter into the diagram?

Principal component model (rank-reduced)

$$\mathbf{x} = \mathbf{A}\mathbf{f} + \mathbf{e}$$

$$\mathrm{cov}(\mathbf{x}) = \mathbf{A}\mathbf{A}' + \mathbf{\Theta}$$



- 2 orthogonal components fitted to 4 manifest variables
- Rotationally indeterminant if no component loadings are constrained (e.g., to zero)

LISREL notation

- Like the standard matrix notation, the LISREL notation uses

  Lower italic: a scalar (as a variable or a parameter; *cf*. X)
  Lower bold: vector
  Upper bold: matrix

- Uses Roman letters only for observed (or manifest, indicator) variables (e.g., $\mathbf{x}$, $\mathbf{y}$) and Greek letters for all others (i.e., latent variables and model parameters; $\boldsymbol{\xi}$, $\boldsymbol{\Gamma}$)

- Distinguishes between exogenous (independent) vs. endogenous (dependent) variables parts

  - We will consider later an alternative notation and model representation --- the reticular action model (RAM, chap. 9)

Notation for latent variables

Symbol	size	Definition
Variables		
$\boldsymbol{\xi}$	$n \times 1$	latent exogenous variables
$\boldsymbol{\eta}$	$m \times 1$	latent endogenous variables
$\boldsymbol{\zeta}$	$m \times 1$	specification error terms
Coefficients		
$\boldsymbol{\Gamma}$	$m \times n$	coefficient matrix for $\boldsymbol{\xi}$
$\mathbf{B}$	$m \times m$	coefficient matrix for $\boldsymbol{\eta}$
Covariance matrix		
$\boldsymbol{\Phi}$	$n \times n$	covariance matrix of $\boldsymbol{\xi}$, $E(\boldsymbol{\xi}\boldsymbol{\xi}')$
$\boldsymbol{\Psi}$	$m \times m$	covariance matrix of $\boldsymbol{\zeta}$, $E(\boldsymbol{\zeta}\boldsymbol{\zeta}')$

Notation for measurement models

Symbol	size	Definition
variables		
$\mathbf{x}$	$q \times 1$	observed indicators of $\boldsymbol{\xi}$
$\mathbf{y}$	$p \times 1$	observed indicators of $\boldsymbol{\eta}$
$\boldsymbol{\delta}$	$q \times 1$	measurement errors for $\mathbf{x}$
$\boldsymbol{\varepsilon}$	$p \times 1$	measurement errors for $\mathbf{y}$
coefficients		
$\boldsymbol{\Lambda}_x$	$q \times n$	loadings relating $\mathbf{x}$ to $\boldsymbol{\xi}$
$\boldsymbol{\Lambda}_y$	$p \times m$	loadings relating $\mathbf{y}$ to $\boldsymbol{\eta}$
covariance matrix		
$\boldsymbol{\Theta}_\delta$	$q \times q$	covariance matrix of $\boldsymbol{\delta}$, $E(\boldsymbol{\delta}\boldsymbol{\delta}')$
$\boldsymbol{\Theta}_\varepsilon$	$p \times p$	covariance matrix of $\boldsymbol{\varepsilon}$, $E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')$

Equation for latent variables

$$\boldsymbol{\eta} = \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\Gamma}\boldsymbol{\xi} + \boldsymbol{\zeta} \quad \text{or} \quad \boldsymbol{\eta} = \left(\mathbf{I} - \mathbf{B}\right)^{-1}\left(\boldsymbol{\Gamma}\boldsymbol{\xi} + \boldsymbol{\zeta}\right)$$

$$\begin{bmatrix} \eta_1 \\ \vdots \\ \eta_m \end{bmatrix} = \begin{bmatrix} 0 & \beta_{12} & \cdots & \beta_{1m} \\ \beta_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \beta_{m-1,m} \\ \beta_{m1} & \cdots & \beta_{m,m-1} & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_m \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & \ddots & \vdots \\ \gamma_{m1} & \cdots & \gamma_{mn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} + \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_m \end{bmatrix}$$

$$\eta_i = \left(\cdots + \beta_{i(i-2)}\eta_{i-2} + \beta_{i(i-1)}\eta_{i-1}\right) + \left(\gamma_{i1}\xi_1 + \gamma_{i2}\xi_2 + \cdots + \gamma_{in}\xi_n\right) + \zeta_i$$

- Equations are <u>linear</u> both in the variables (η, ξ) and in the parameters (β, γ); same for equations for manifest variables

Assumptions

- $E(\boldsymbol{\eta}) = \mathbf{0}, \quad E(\boldsymbol{\xi}) = \mathbf{0}, \quad E(\boldsymbol{\zeta}) = \mathbf{0}$
- $E(\boldsymbol{\zeta}\boldsymbol{\xi}') = \mathbf{0}_{m \times n}$
- $(\mathbf{I} - \mathbf{B})$ is nonsigular so that $(\mathbf{I} - \mathbf{B})^{-1}$ exists
- ζ_i has a homogeneous variance for all subjects, i.e.,

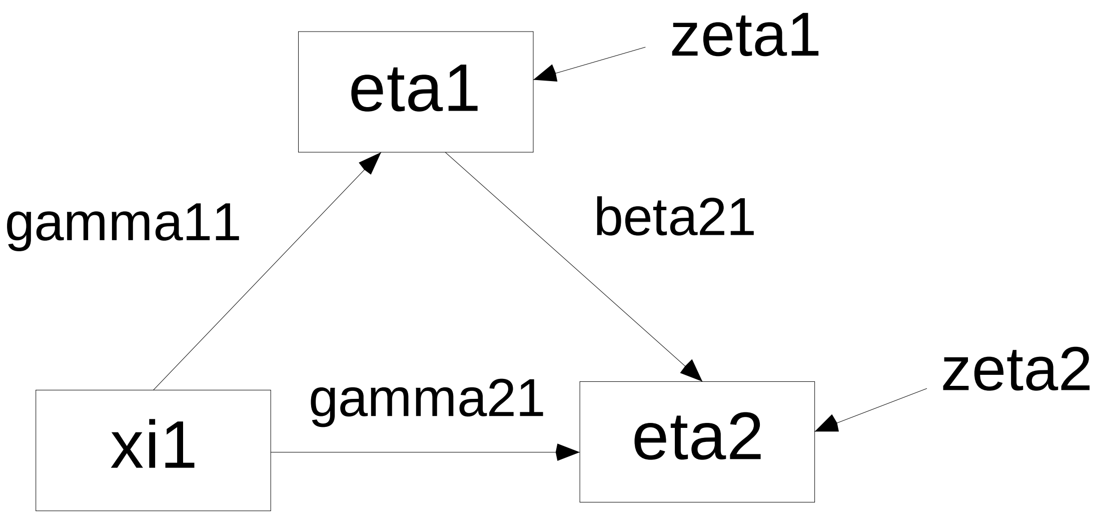
  $$E(\zeta_{ik}^2) = \mathrm{var}(\zeta_i) \text{ for } k = 1, \ldots, N, \; i = 1, \ldots, m$$

  and independent, i.e., $\mathrm{cov}(\zeta_{ik}, \zeta_{il}) = 0$ for all $k \neq l$;
  otherwise, multilevel structure

An example

$$\boldsymbol{\eta} = \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\Gamma}\boldsymbol{\xi} + \boldsymbol{\zeta}$$

$$\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \beta_{21} & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} \gamma_{11} \\ \gamma_{21} \end{bmatrix} \xi_1 + \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$$



Measurement models

$$\mathbf{x} = \mathbf{\Lambda}_x \boldsymbol{\xi} + \boldsymbol{\delta}$$

$$\mathbf{y} = \mathbf{\Lambda}_y \boldsymbol{\eta} + \boldsymbol{\varepsilon}$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_q \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{q1} & \cdots & \lambda_{qn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} + \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_q \end{bmatrix}$$

$$x_i = \lambda_{i1}\xi_1 + \lambda_{i2}\xi_2 + \cdots + \lambda_{in}\xi_n + \delta_i$$

- Often, a uni-factorial structure is imposed on $\mathbf{\Lambda}_x$, $\mathbf{\Lambda}_y$ such as:

$$\mathbf{\Lambda}_x = \begin{bmatrix} \lambda_{11} & 0 & 0 \\ \lambda_{21} & 0 & 0 \\ \lambda_{31} & 0 & 0 \\ 0 & \lambda_{42} & 0 \\ 0 & \lambda_{52} & 0 \\ 0 & \lambda_{62} & 0 \\ 0 & 0 & \lambda_{73} \\ 0 & 0 & \lambda_{83} \\ 0 & 0 & \lambda_{93} \end{bmatrix}$$

Assumptions

- All latent variables expected to be zero --- a natural consequence when fitting mean-centered manifest variables

$$E(\boldsymbol{\eta}) = \mathbf{0}, \quad E(\boldsymbol{\xi}) = \mathbf{0}, \quad E(\boldsymbol{\varepsilon}) = \mathbf{0}, \quad E(\boldsymbol{\delta}) = \mathbf{0}$$

But not so natural when multiple groups or hierarchically nested (multilevel) data are considered

- Observed variables are correlated only through the modeled latent variables and parameters (latent path coefficients and loadings), with simplifying conditions of

$$E(\boldsymbol{\varepsilon}\boldsymbol{\eta}') = \mathbf{0}, \quad E(\boldsymbol{\varepsilon}\boldsymbol{\xi}') = \mathbf{0}, \quad E(\boldsymbol{\varepsilon}\boldsymbol{\zeta}') = \mathbf{0}, \quad E(\boldsymbol{\varepsilon}\boldsymbol{\delta}') = \mathbf{0}$$

$$E(\boldsymbol{\delta}\boldsymbol{\xi}') = \mathbf{0}, \quad E(\boldsymbol{\delta}\boldsymbol{\eta}') = \mathbf{0}, \quad E(\boldsymbol{\delta}\boldsymbol{\zeta}') = \mathbf{0}$$

- Typically, $\mathbf{\Theta}_\delta$ and $\mathbf{\Theta}_\varepsilon$ are considered to be diagonal matrices, i.e., measurement errors of the indicators are uncorrelated; however, there may be justifiable cases for non-diagonal $\mathbf{\Theta}_\delta$ and $\mathbf{\Theta}_\varepsilon$ or even $E(\boldsymbol{\varepsilon}\boldsymbol{\delta}') \neq \mathbf{0}$, for some selective entries

  e.g., Fig. 2.6 (p. 37) of industrialization and democracy model,

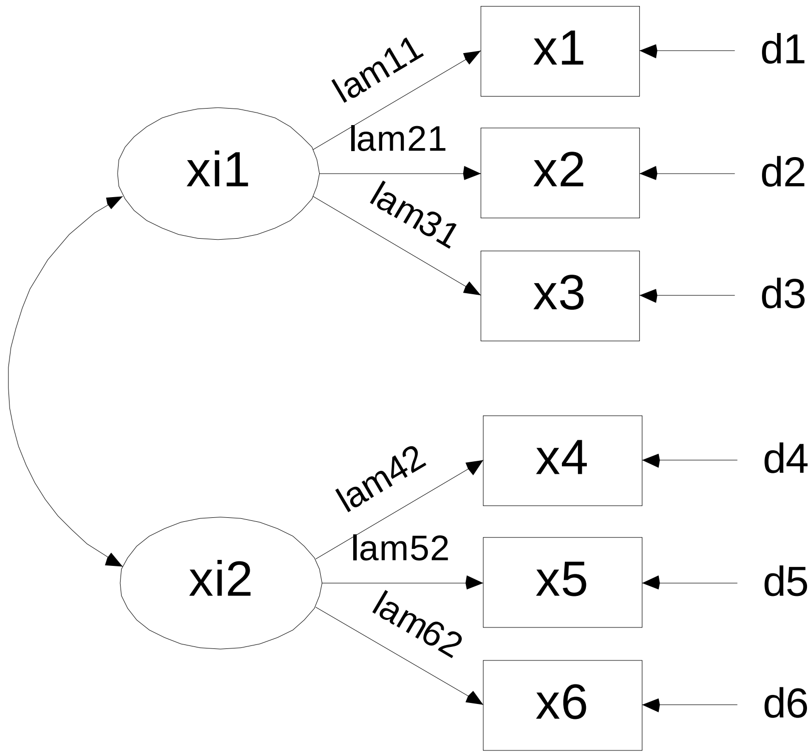
- Like ζ, δ and ε are assumed to be homoscedestic and independent (i.e., iid)

- We will consider later distributional assumptions for manifest variables $\mathbf{x}$ and $\mathbf{y}$

An example

$$\mathbf{x} = \mathbf{\Lambda}_x \boldsymbol{\xi} + \boldsymbol{\delta}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \lambda_{11} & 0 \\ \lambda_{21} & 0 \\ \lambda_{31} & 0 \\ 0 & \lambda_{42} \\ 0 & \lambda_{52} \\ 0 & \lambda_{62} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{bmatrix}$$



- Where is the double-headed arrow defined in the equation?

Algebraic rules for covariance

- Following rules may be useful:

$$\operatorname{cov}(c, X) = 0$$

$$\operatorname{cov}(cX_1, X_2) = c \cdot \operatorname{cov}(X_1, X_2)$$

$$\operatorname{cov}(X_1 + X_2, X_3) = \operatorname{cov}(X_1, X_3) + \operatorname{cov}(X_2, X_3)$$

- But the standard algebra will do; suppose, e.g., $x_1 = \lambda_1 \xi_1 + \delta_1$, $x_2 = \lambda_2 \xi_1 + \delta_2$:

$$\operatorname{cov}(x_1, x_2) = E(x_1 x_2) = E(\lambda_1 \xi_1 + \delta_1)(\lambda_2 \xi_1 + \delta_2)$$

$$= E(\lambda_1 \lambda_2 \xi_1 \xi_1 + \lambda_1 \delta_2 \xi_1 + \lambda_2 \delta_1 \xi_1 + \delta_1 \delta_2)$$

$$= \lambda_1 \lambda_2 E(\xi_1 \xi_1) = \lambda_1 \lambda_2 \phi_{11}$$

- Matrix algebra useful for operation of covariance matrices:

$$\mathbf{x} = \mathbf{\Lambda}_x \boldsymbol{\xi} + \boldsymbol{\delta},$$

$$\begin{aligned}
\operatorname{cov}(\mathbf{x}, \mathbf{x}) = E(\mathbf{x}\mathbf{x}') &= E(\mathbf{\Lambda}_x \boldsymbol{\xi} + \boldsymbol{\delta})(\boldsymbol{\xi}'\mathbf{\Lambda}'_x + \boldsymbol{\delta}') \\
&= E(\mathbf{\Lambda}_x \boldsymbol{\xi}\boldsymbol{\xi}'\mathbf{\Lambda}'_x + \mathbf{\Lambda}_x \boldsymbol{\xi}\boldsymbol{\delta}' + \boldsymbol{\delta}\boldsymbol{\xi}'\mathbf{\Lambda}'_x + \boldsymbol{\delta}\boldsymbol{\delta}') \\
&= \mathbf{\Lambda}_x E(\boldsymbol{\xi}\boldsymbol{\xi}')\mathbf{\Lambda}'_x + E(\boldsymbol{\delta}\boldsymbol{\delta}') = \mathbf{\Lambda}_x \mathbf{\Phi}\mathbf{\Lambda}'_x + \mathbf{\Theta}_\delta
\end{aligned}$$

Effect decomposition

- Direct effect: unmediated expected change on a variable due to another --- a path coefficient

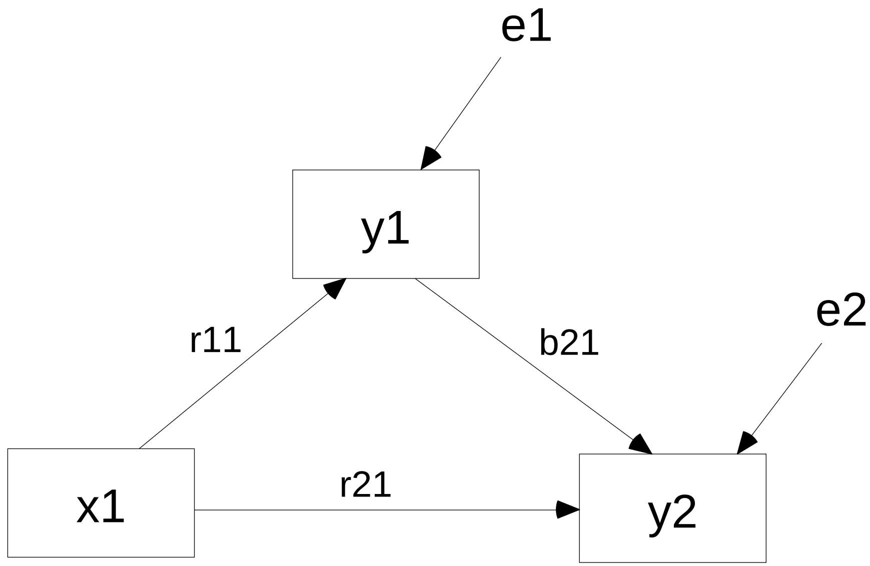
- Indirect effect: all other possible influences from a variable to another, other than its direct effect --- product of all involved mediating path coefficients for each indirect effect

- Total effect = direct effect + all indirect effects

- Recursive vs. non-recursive models

- convergent vs. divergent series of indirect effects

Effect decomposition example 1

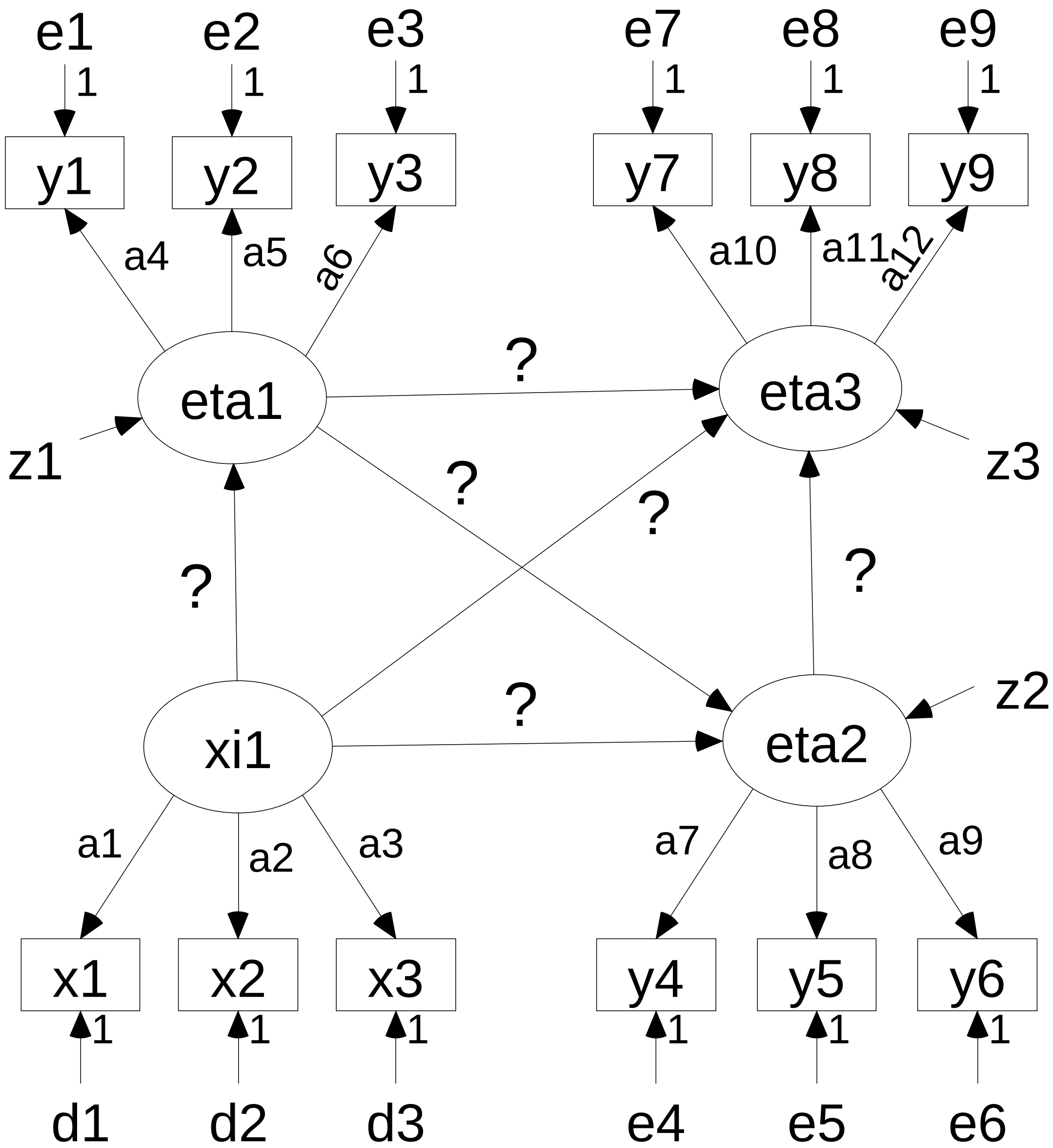


- Direct effect of x_1 on $y_2 = \gamma_{21}$

  Indirect effect $= \gamma_{11}\beta_{21}$ --- expected change on y_2 due to 1 unit change on x_1

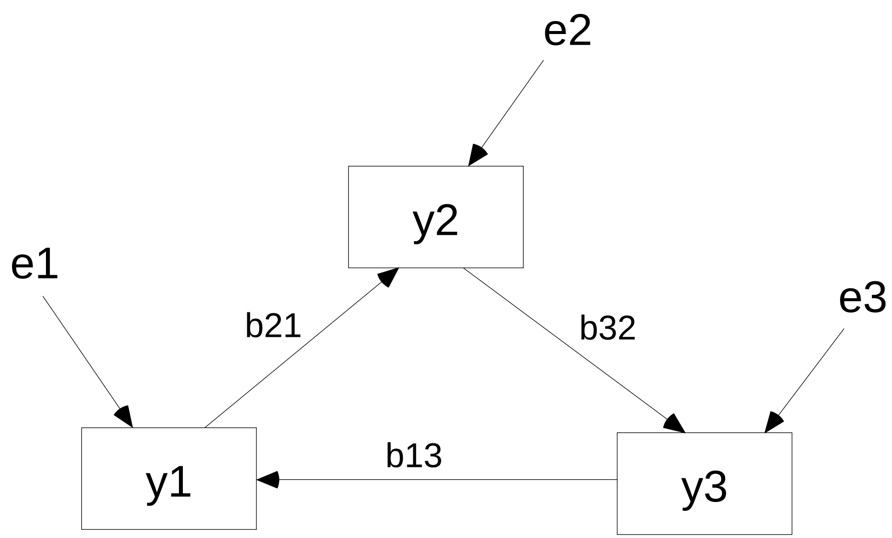
  Total effect $= \gamma_{21} + \gamma_{11}\beta_{21}$

Effect decomposition example 2



Effect decomposition example 3

- non-recursive since all 3 variables indirectly influence itself



$$\mathbf{B} = \begin{bmatrix} 0 & 0 & \beta_{13} \\ \beta_{21} & 0 & 0 \\ 0 & \beta_{32} & 0 \end{bmatrix}$$

- Indirect effect of y_1 on y_3:

$$\beta_{21}\beta_{32} + \beta_{21}\beta_{32}\left(\beta_{13}\beta_{21}\beta_{32}\right) + \beta_{21}\beta_{32}\left(\beta_{13}\beta_{21}\beta_{32}\right)^2 + \cdots$$

$$= \beta_{21}\beta_{32}\sum_{k=0}^{\infty}\left(\beta_{13}\beta_{21}\beta_{32}\right)^k$$