Confirmatory Factor Analysis: measurement models

Psychology 588: Covariance structure and factor models

Few things on measurement theory

- Measurement may be defined as assigning numbers to observational units (“subjects”) so as to indicate the magnitude (and direction if bipolar) on an intended concept (construct, latent variable)

- A measurement model quantitatively defines the relationships between observed phenomena (items, indicators, measures, manifest variables) and unobservable concepts (factors, latent variables)

  - e.g., Spearman’s single factor model, 2-parameter logistic model, graded response model, etc.

  - We will only consider linear relationships in SEM (in the classical form, while generalized SEM allows nonlinear relationships)

Spearman's single factor model

- A set of measures are modeled to linearly relate to a single common factor as:

$$x_i = \lambda_i \xi + \delta_i \qquad \mathbf{x} = \boldsymbol{\lambda} \xi + \boldsymbol{\delta}$$

- With means included:

$$x_i = \nu_i + \lambda_i \xi + \delta_i, \quad E\left(\xi\right) = \kappa, \quad E\left(\delta_i\right) = 0$$

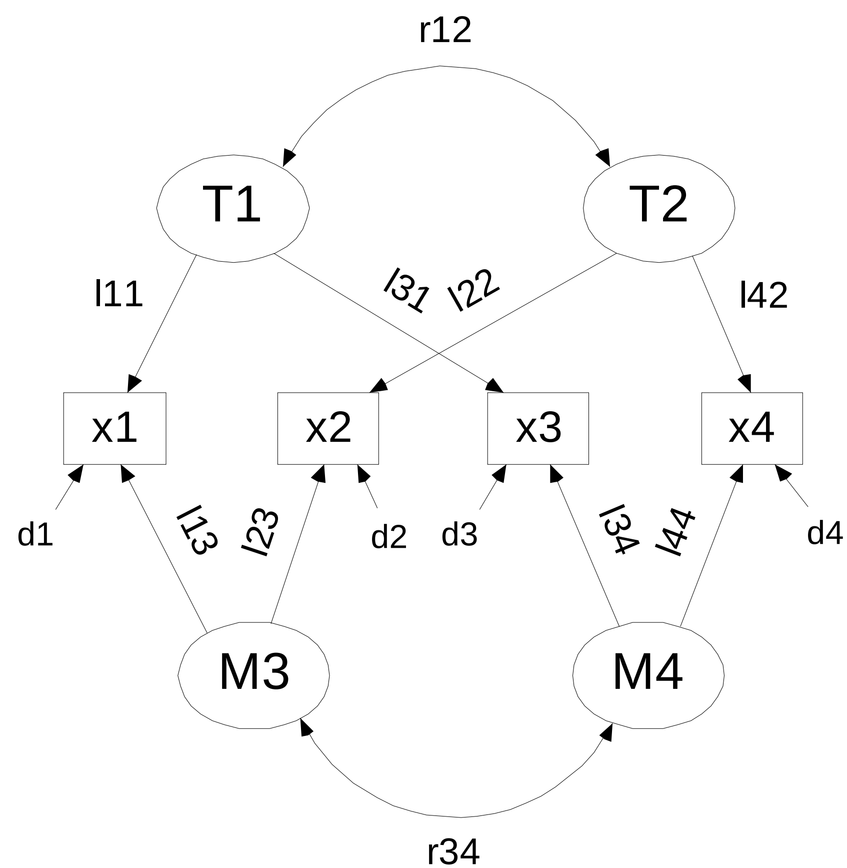
$$\mathbf{x} = \boldsymbol{\nu} + \boldsymbol{\lambda} \xi + \boldsymbol{\delta}$$

- Rescaling does (and should) not affect measurement structure --- though it's not trivial to find right rescaling, e.g., in multiple group analysis

Convergent & discriminant validity

- According to construct validity, a properly measured latent variable should strongly covary with an item that is theoretically believed to highly relate with; and should weakly covary with one that is theoretically believed to relate not so much with

- MTMM (multitrait-multimethod) procedure is one modeling technique to verify a type of convergent and discriminant validity (Fig 6.4 in p.192; also see Tables 6.2 and 6.3)



$$\rho_{x_1x_3} = \lambda_{11}\lambda_{31} + \lambda_{13}\lambda_{34}\rho_{\xi_3\xi_4}$$

$$\rho_{x_1x_2} = \lambda_{13}\lambda_{23} + \lambda_{11}\lambda_{22}\rho_{\xi_1\xi_2}$$

Validity considered in SEM

- (Construct, criterion, convergent-divergent, etc.) validity is typically assessed by bivariate correlations of observed variables --- possibly misleading due to measurement errors

- Alternatively, validity of a measure x_i of ξ_j may be defined by the magnitude of the direct structural relation between ξ_j and x_i

  - Unstandardized validity coefficient: λ_{ij}

  - Standardized validity coefficient:

$$\lambda_{ij}^{(\mathrm{s})} = \lambda_{ij} \left(\frac{\phi_{jj}}{\mathrm{var}(x_i)} \right)^{0.5}$$

- Unique validity variance (analogous to incremental R^2 of x_i solely due to ξ_j):

$$U_{x_i\xi_j} = R^2_{x_i} - R^2_{x_i(\sim\xi_j)}$$

where the SMCs are generally given by:

$$R^2_{x_i} = \frac{\boldsymbol{\sigma}'_{x_i\xi}\tilde{\boldsymbol{\Phi}}^{-1}\boldsymbol{\sigma}_{x_i\xi}}{\operatorname{var}(x_i)}, \qquad R^2_{x_i(\sim\xi_j)} = \frac{\boldsymbol{\sigma}'_{x_i(\sim\xi_j)}\tilde{\boldsymbol{\Phi}}^{-1}_{(\sim\xi_j)}\boldsymbol{\sigma}_{x_i(\sim\xi_j)}}{\operatorname{var}(x_i)}$$

where $\tilde{\boldsymbol{\Phi}}$ is a submatrix of $\boldsymbol{\Phi}$ only including the $\boldsymbol{\xi}$ that directly influence x_i; $\sim\xi_j$ indicates all $\boldsymbol{\xi}$ in $\tilde{\boldsymbol{\Phi}}$ except the j-th; and $\boldsymbol{\sigma}_{x_i\xi}$ is a vector of covariances between x_i and all ξ in $\tilde{\boldsymbol{\Phi}}$

- Note that $R^2_{x_i} = \boldsymbol{\lambda}'_i\boldsymbol{\Phi}\boldsymbol{\lambda}_i / \operatorname{var}(x_i)$ with $\boldsymbol{\sigma}'_{x_i\boldsymbol{\xi}} = \boldsymbol{\lambda}'_i\boldsymbol{\Phi}$

Classical test-theory models viewed from SEM

$$x_i = \alpha_i \tau + e_i, \qquad x_j = \alpha_j \tau + e_j$$

$$E(\tau e) = 0, \quad E(e) = 0, \quad E(e_i e_j) = 0$$

- Parallel measures: $\alpha_i = \alpha_j = 1, \quad \mathrm{var}(e_i) = \mathrm{var}(e_j)$
- tau-equivalent measures: $\alpha_i = \alpha_j = 1, \quad \mathrm{var}(e_i) \neq \mathrm{var}(e_j)$
- congeneric measures: $\alpha_i \neq \alpha_j, \quad \mathrm{var}(e_i) \neq \mathrm{var}(e_j)$
  - essentially equivalent to the uni-factorial measurement models in SEM

Joreskog, K.G. (1971). Statistical analysis of sets of congeneric tests. *Psychometrika, 36*, 109-133.

- Internal-consistency reliability is the ratio of the variance due to true scores to the variance of observed variable, which equals the squared correlation between x_i and τ (i.e., $\rho^2_{x_i\tau}$)

$$\rho_{ii} = \frac{\alpha_i^2 \operatorname{var}(\tau)}{\operatorname{var}(x_i)} = \frac{\operatorname{cov}(x_i, \tau)^2}{\operatorname{var}(x_i)\operatorname{var}(\tau)} = \rho^2_{x_i\tau}$$

$$0 \le \rho_{ii} \le 1$$

- Reliability can be assessed in several ways: test-retest, split-half, Coefficient α --- the latter assumes measures to be parallel or tau-equivalent and, consequently, it’s a lower bound of the reliability measured as internal consistency

- Given a set of congeneric measures, the reliability of their unweighted sum (as scale scores) is:

$$\rho_{HH} = \frac{\left(\sum_{i=1}^{q} \alpha_i\right)^2 \mathrm{var}(\tau)}{\mathrm{var}(x_H)}, \qquad x_H = \sum_{i=1}^{q} x_i$$

which reduces to $\rho_{HH} = q^2 \mathrm{var}(\tau) / \mathrm{var}(x_H)$ for parallel or tau-equivalent measures from which the formula for Coefficient α can be derived as (see Eq. 6.40, p. 216)

$$\rho_{H\tau}^2 = \rho_{HH} \geq \frac{q}{q-1}\left(1 - \frac{\sum_{i=1}^{q} \mathrm{var}(x_i)}{\mathrm{var}(x_H)}\right) \equiv \alpha$$

- Cronbach, L.J. (1951). Coefficent alpha and the internal structure of tests. *Psychometrika, 16*, 297-334.
- Gulliksen, H. (1950). *Theory of mental tests*. New York: Wiley, 1950.
- Guttman, L. (1945). A basis for analyzing test-retest reliability. *Psychometrika, 10*, 255-282.

Exploratory vs. confirmatory FA

- Exploratory-confirmatory distinction is better made on a continuum rather than by a strict dichotomy --- people do an exploratory analysis with “CFA programs” (e.g., AMOS) and a confirmatory analysis with “EFA programs” (e.g., “data reduction” in SPSS)

- Both EFA and CFA take the same model form, while they have different ways of imposing constraints --- EFA with minimal constraints for identification and CFA with further constraints, typically at specific entries in the loading matrix

- CFA sometimes refers to “Common Factor Analysis” as opposed to “Principal Component Analysis” (PCA)

Common to both EFA and CFA

- Linear relationship between factors and indicators
- Error terms not correlated with factors
- No distinction of cause and effect among indicators (unlike regression)
- Many dimensional observed variables (indicators) approximated by less dimensional latent variables (factors)
- Subject to scaling indeterminacy

Differences between EFA and CFA

- EFA is indeterminant in dimensionality and rotation --- CFA is determinant (i.e., all parameters are identifiable) thanks to selective constraints on $\mathbf{\Lambda}$

  Rotational indeterminacy in EFA is solved by seeking an optimal rotation to a "simple structure"

- EFA finds an *R*-dimensional solution by which covariances between variables are explainable parsimoniously --- CFA imposes a specific loading pattern and tests how bad fit it results in

- EFA tends to rely more on the data --- CFA needs a "good" understanding of the factor structure (that wouldn't vary over samples)

Specification

- Same equation for CFA and EFA:

$$\mathbf{x} = \boldsymbol{\Lambda}_x \boldsymbol{\xi} + \boldsymbol{\delta} \quad \left(\mathbf{y} = \boldsymbol{\Lambda}_y \boldsymbol{\eta} + \boldsymbol{\varepsilon}\right)$$

- Only difference is whether we selectively put some constraints on $\boldsymbol{\Lambda}$ and $E\left(\boldsymbol{\xi}\boldsymbol{\xi}'\right)$, and some relaxation on $E\left(\boldsymbol{\delta}\boldsymbol{\delta}'\right)$

- Further partitioning of measurement errors: $\boldsymbol{\delta} = \mathbf{s} + \mathbf{e}$

$\mathbf{s}$ --- specific variance, not shared with any other indicators but replicable over random samples

$\mathbf{e}$ --- remaining random variance, not replicable

Note: $\boldsymbol{\delta}$ is typically called “unique factor” in factor analysis

EFA vs. CFA rethought

- About the political democracy example (pp. 231 & 235),
  - What if most of covariation among x_i explainable by two distinct aspects of democracy which do not vary much between 1960 and 1965?
  - Are the equality constraints and correlated δ_i (particularly, the correlated δ_2-δ_4 and δ_6-δ_8 pairs) shown in Fig 7.3 more sensible than EFA in Fig 7.2?
  - Would it be like this example for every factor-analysis application?

Model specification in CFA

- Only one structural equation is sufficient since all factors are exogenous variables:

$$\mathbf{x} = \mathbf{\Lambda}_x \boldsymbol{\xi} + \boldsymbol{\delta}$$

- Although it's typical to have only one free parameter per row in $\mathbf{\Lambda}_x$ (i.e., uni-factorial pattern), such "extreme" constraints are not necessary provided that whatever less constrained model is identifiable

- Also, flexible constraints (and/or relaxation) can be imposed on off-diagonal entries of $\mathbf{\Phi}$ and $\mathbf{\Theta}_\delta$

- In theory, constraints can be zero, non-zero constant, equality, or inequality --- "can estimate?" is another question

Implied covariance matrix

- Covariance structure of measurement model:

$$\boldsymbol{\Sigma}(\boldsymbol{\theta}) = E(\mathbf{x}\mathbf{x}') = E(\boldsymbol{\Lambda}_x\boldsymbol{\xi} + \boldsymbol{\delta})(\boldsymbol{\xi}'\boldsymbol{\Lambda}_x' + \boldsymbol{\delta}')$$

$$= \boldsymbol{\Lambda}_x E(\boldsymbol{\xi}\boldsymbol{\xi}')\boldsymbol{\Lambda}_x' + E(\boldsymbol{\delta}\boldsymbol{\delta}')$$

$$= \boldsymbol{\Lambda}_x \boldsymbol{\Phi} \boldsymbol{\Lambda}_x' + \boldsymbol{\Theta}_\delta$$

With constraints, many individual parameters will vanish --- particularly so with the uni-factorial loading pattern