Confirmatory Factor Analysis: measurement models
Few things on measurement theory

• Measurement may be defined as assigning numbers to observational units (“subjects”) so as to indicate the magnitude (and direction if bipolar) on an intended concept (construct, latent variable)

• A measurement model quantitatively defines the relationships between observed phenomena (items, indicators, measures, manifest variables) and unobservable concepts (factors, latent variables)

  ➢ e.g., Spearman’s single factor model, 2-parameter logistic model, graded response model, etc.

  ➢ We will only consider linear relationships in SEM (in the classical form, while generalized SEM allows nonlinear relationships)
Spearman’s single factor model

- A set of measures are modeled to linearly relate to a single common factor as:

\[ x_i = \lambda_i \xi + \delta_i \quad x = \lambda \xi + \delta \]

- With means included:

\[ x_i = \nu_i + \lambda_i \xi + \delta_i, \quad E(\xi) = \kappa, \quad E(\delta_i) = 0 \]

\[ x = \nu + \lambda \xi + \delta \]

- Rescaling does (and should) not affect measurement structure --- though it’s not trivial to find right rescaling, e.g., in multiple group analysis
Convergent & discriminant validity

- According to construct validity, a properly measured latent variable should strongly covary with an item that is theoretically believed to highly relate with; and should weakly covary with one that is theoretically believed to relate not so much with.

- MTMM (multitrait-multimethod) procedure is one modeling technique to verify a type of convergent and discriminant validity (Fig 6.4 in p.192; also see Tables 6.2 and 6.3)

\[
\rho_{x_1x_3} = \lambda_{11}\lambda_{31} + \lambda_{13}\lambda_{34}\rho_{\xi_3\xi_4}
\]

\[
\rho_{x_1x_2} = \lambda_{13}\lambda_{23} + \lambda_{11}\lambda_{22}\rho_{\xi_1\xi_2}
\]
• (Construct, criterion, convergent-divergent, etc.) validity is typically assessed by bivariate correlations of observed variables --- possibly misleading due to measurement errors.

• Alternatively, validity of a measure $x_i$ of $\xi_j$ may be defined by the magnitude of the direct structural relation between $\xi_j$ and $x_i$.

  - Unstandardized validity coefficient: $\lambda_{ij}$
  - Standardized validity coefficient:

$$\lambda_{ij}^{(s)} = \lambda_{ij} \left( \frac{\phi_{jj}}{\text{var}(x_i)} \right)^{0.5}$$
Unique validity variance (analogous to incremental $R^2$ of $x_i$ solely due to $\xi_j$):

$$U_{x_i\xi_j} = R^2_{x_i} - R^2_{x_i(\sim\xi_j)}$$

where the SMCs are generally given by:

$$R^2_{x_i} = \frac{\sigma'_{x_i\xi} \tilde{\Phi}^{-1} \sigma_{x_i\xi}}{\text{var}(x_i)}, \quad R^2_{x_i(\sim\xi_j)} = \frac{\sigma'_{x_i(\sim\xi_j)\xi_{j}} \tilde{\Phi}^{-1}_{(\sim\xi_j)} \sigma_{x_i(\sim\xi_j)}}{\text{var}(x_i)}$$

where $\tilde{\Phi}$ is a submatrix of $\Phi$ only including the $\xi$ that directly influence $x_i$; $\sim\xi_j$ indicates all $\xi$ in $\tilde{\Phi}$ except the $j$-th; and $\sigma_{x_i\xi}$ is a vector of covariances between $x_i$ and all $\xi$ in $\tilde{\Phi}$.

Note that $R^2_{x_i} = \lambda'_i \Phi \lambda_i / \text{var}(x_i)$ with $\sigma'_{x_i\xi} = \lambda'_i \Phi$.
Classical test-theory models viewed from SEM

\[ x_i = \alpha_i \tau + e_i, \quad x_j = \alpha_j \tau + e_j \]

\[ E(\tau e) = 0, \quad E(e) = 0, \quad E(e_i e_j) = 0 \]

- Parallel measures: \( \alpha_i = \alpha_j = 1, \quad \text{var}(e_i) = \text{var}(e_j) \)
- Tau-equivalent measures: \( \alpha_i = \alpha_j = 1, \quad \text{var}(e_i) \neq \text{var}(e_j) \)
- Congeneric measures: \( \alpha_i \neq \alpha_j, \quad \text{var}(e_i) \neq \text{var}(e_j) \)

- essentially equivalent to the uni-factorial measurement models in SEM

• Internal-consistency reliability is the ratio of the variance due to true scores to the variance of observed variable, which equals the squared correlation between $x_i$ and $\tau$ (i.e., $\rho_{x_i\tau}^2$)

$$\rho_{ii} = \frac{\alpha_i^2 \text{var}(\tau)}{\text{var}(x_i)} = \frac{\text{cov}(x_i, \tau)^2}{\text{var}(x_i) \text{var}(\tau)} = \rho_{x_i\tau}^2$$

$$0 \leq \rho_{ii} \leq 1$$

• Reliability can be assessed in several ways: test-retest, split-half, Coefficient $\alpha$ --- the latter assumes measures to be parallel or tau-equivalent and, consequently, it’s a lower bound of the reliability measured as internal consistency
• Given a set of congeneric measures, the reliability of their unweighted sum (as scale scores) is:

\[
\rho_{HH} = \frac{\left(\sum_{i=1}^{q} \alpha_i\right)^2 \text{var}(\tau)}{\text{var}(x_H)}, \quad x_H = \sum_{i=1}^{q} x_i
\]

which reduces to \( \rho_{HH} = q^2 \frac{\text{var}(\tau)}{\text{var}(x_H)} \) for parallel or tau-equivalent measures from which the formula for Coefficient \( \alpha \) can be derived as (see Eq. 6.40, p. 216)

\[
\rho_{H\tau}^2 = \rho_{HH} \geq \frac{q}{q-1} \left(1 - \sum_{i=1}^{q} \frac{\text{var}(x_i)}{\text{var}(x_H)}\right) \equiv \alpha
\]

Exploratory vs. confirmatory FA

• Exploratory-confirmatory distinction is better made on a continuum rather than by a strict dichotomy --- people do an exploratory analysis with “CFA programs” (e.g., AMOS) and a confirmatory analysis with “EFA programs” (e.g., “data reduction” in SPSS)

• Both EFA and CFA take the same model form, while they have different ways of imposing constraints --- EFA with minimal constraints for identification and CFA with further constraints, typically at specific entries in the loading matrix

• CFA sometimes refers to “Common Factor Analysis” as opposed to “Principal Component Analysis” (PCA)
Common to both EFA and CFA

- Linear relationship between factors and indicators
- Error terms not correlated with factors
- No distinction of cause and effect among indicators (unlike regression)
- Many dimensional observed variables (indicators) approximated by less dimensional latent variables (factors)
- Subject to scaling indeterminacy
Differences between EFA and CFA

- EFA is indeterminant in dimensionality and rotation --- CFA is determinant (i.e., all parameters are identifiable) thanks to selective constraints on $\Lambda$

  Rotational indeterminacy in EFA is solved by seeking an optimal rotation to a “simple structure”

- EFA finds an $R$-dimensional solution by which covariances between variables are explainable parsimoniously --- CFA imposes a specific loading pattern and tests how bad fit it results in

- EFA tends to rely more on the data --- CFA needs a “good” understanding of the factor structure (that wouldn’t vary over samples)
Same equation for CFA and EFA:

\[ x = \Lambda_x \xi + \delta \quad \left( y = \Lambda_y \eta + \varepsilon \right) \]

Only difference is whether we selectively put some constraints on \( \Lambda \) and \( E(\xi\xi') \), and some relaxation on \( E(\delta\delta') \).

Further partitioning of measurement errors:

\[ \delta = s + e \]

- \( s \) --- specific variance, not shared with any other indicators but replicable over random samples
- \( e \) --- remaining random variance, not replicable

Note: \( \delta \) is typically called “unique factor” in factor analysis.
EFA vs. CFA rethought

- About the political democracy example (pp. 231 & 235),

  - What if most of covariation among $x_i$ explainable by two distinct aspects of democracy which do not vary much between 1960 and 1965?

  - Are the equality constraints and correlated $\delta_i$ (particularly, the correlated $\delta_2$-$\delta_4$ and $\delta_6$-$\delta_8$ pairs) shown in Fig 7.3 more sensible than EFA in Fig 7.2?

  - Would it be like this example for every factor-analysis application?
Model specification in CFA

• Only one structural equation is sufficient since all factors are exogenous variables:

\[ x = \Lambda_x \xi + \delta \]

• Although it’s typical to have only one free parameter per row in \( \Lambda_x \) (i.e., uni-factorial pattern), such “extreme” constraints are not necessary provided that whatever less constrained model is identifiable

• Also, flexible constraints (and/or relaxation) can be imposed on off-diagonal entries of \( \Phi \) and \( \Theta_\delta \)

• In theory, constraints can be zero, non-zero constant, equality, or inequality --- “can estimate?” is another question
Implied covariance matrix

- Covariance structure of measurement model:

\[
\Sigma(\theta) = E(xx') = E(\Lambda_x \xi + \delta)(\xi'\Lambda_x' + \delta') = \Lambda_x E(\xi\xi') \Lambda_x' + E(\delta\delta') = \Lambda_x \Phi \Lambda_x' + \Theta_\delta
\]

With constraints, many individual parameters will vanish --- particularly so with the uni-factorial loading pattern