Confirmatory Factor Analysis: Model comparison, respecification, and more

Model comparison

- Essentially all goodness of fit indices are descriptive, with no statistical device for selecting from alternative models (see table 7.8, p. 290 for the political democracy example)

- Same for other types of fits (e.g., AIC, BIC) or cross-validation technique

- Chi-square difference test available for comparing a nested model with a nesting model, provided that all assumptions are reasonably met and more importantly the nesting model is correct

- Why does a nested model <u>must</u> produce an equal or higher chi-square value regardless of types of constraints (e.g., constant, equality, or any functional form)? Impossible at all to have a lower value?

Likelihood ratio test for ML

$$F_{\mathrm{LR}} = -2\left[\log L\left(\hat{\boldsymbol{\theta}}_{\mathrm{nested}}\right) - \log L\left(\hat{\boldsymbol{\theta}}_{\mathrm{nesting}}\right)\right] = -2\log\frac{L\left(\hat{\boldsymbol{\theta}}_{\mathrm{nested}}\right)}{L\left(\hat{\boldsymbol{\theta}}_{\mathrm{nesting}}\right)}$$

$$= (N-1)\left(F_{\mathrm{nested}} - F_{\mathrm{nesting}}\right) = \chi^2_{\mathrm{nested}} - \chi^2_{\mathrm{nesting}}$$

- F_{LR} itself is a chi-square variable with $df = df_{\mathrm{nested}} - df_{\mathrm{nesting}}$
- Null hypothesis --- a set of constraints (as the only difference between the nested and the nesting model) hold in the population
- F_{LR} is conditional to the nesting model --- consequence of an additional constraint will depend on what’s already imposed, e.g., significance for pairs of $F_1 > F_2 > F_3 > F_4$ are not necessarily consistent with the order

- LR test is tedious when we want to find a statistically justifiable “best” fitting model with respect to a set of meaningful constraints; or put differently, when we want to explore for a most optimal model among many alternative, substantively justifiable models

- Now we need a method that allows for statistical inference about:

  - What if a set of constraints in a given model is freed?

  - What if a set of freely estimated parameters are constrained?

Lagrangian Multiplier test

- LM test answers "What if a set of constraints are freed?" only based on estimates of a nested (more restricted) model

- What's suggested by LM is the expectation of chi-square change (and the associated parameter estimates) if some constraints are removed --- tends to underestimate the chi-square reduction compared to the difference by LR test

- When only one constraint is considered, LM is called "modification index" (which is available in most SEM programs including AMOS) --- though the LM statistic is defined for any subset of the current constraints, SEM programs print only LM for each constraint

- Consider a set of constrained parameters $\boldsymbol{\theta}_0$ (not necessarily all zero) for $\boldsymbol{\theta}_\mathrm{a}$ in a partitioned set, $\boldsymbol{\theta} = [\boldsymbol{\theta}'_\mathrm{a}, \boldsymbol{\theta}'_\mathrm{b}]'$; then the restricted and unrestricted parameter sets are written, respectively, $\boldsymbol{\theta}_\mathrm{r} = [\boldsymbol{\theta}'_0, \boldsymbol{\theta}'_\mathrm{b}]'$ and $\boldsymbol{\theta}_\mathrm{u} = [\boldsymbol{\theta}'_1, \boldsymbol{\theta}'_\mathrm{b}]'$, where $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_\mathrm{b}$ are freely estimated --- we will use this representation when considering power of testing

- The LM statistic is:

$$F_{\mathrm{LM}} = \mathbf{s}\left(\hat{\boldsymbol{\theta}}_\mathrm{r}\right)' \mathrm{acov}\left(\hat{\boldsymbol{\theta}}_\mathrm{r}\right) \mathbf{s}\left(\hat{\boldsymbol{\theta}}_\mathrm{r}\right)$$

where $\mathbf{s}\left(\hat{\boldsymbol{\theta}}_\mathrm{r}\right)$ is a first-order partial derivatives of an optimization function (e.g., F_{ML}) evaluated at $\hat{\boldsymbol{\theta}}_\mathrm{r}$, and then F_{LM} is chi-square distributed with $df = \#(\boldsymbol{\theta}_0)$; and so by F_{LM} we can tell how much of chi-square improvement to expect due to removing the constraints $\boldsymbol{\theta}_0$

Wald test

- Only by fitting the nesting (less restricted) model, the Wald test answers "What if a set of freely estimated parameters are constrained?"

- The Wald statistic F_{W} is defined as follows and chi-square distributed with $df = \#(\boldsymbol{\theta}_0)$ under H_0 (i.e., $\boldsymbol{\theta}_{\mathrm{a}} = \boldsymbol{\theta}_0$)

$$F_{\mathrm{W}} = \left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_0\right)' \left[\mathrm{acov}\left(\hat{\boldsymbol{\theta}}_1\right)\right]^{-1} \left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_0\right)$$

  where $\mathrm{acov}\left(\hat{\boldsymbol{\theta}}_1\right)$ is an estimate of asymptotic covariance matrix of $\boldsymbol{\theta}_{\mathrm{a}}$ (evaluated at $\hat{\boldsymbol{\theta}}_1$) --- and so a significant F_{W} indicates the constraints being incorrect, $\boldsymbol{\theta}_{\mathrm{a}} \neq \boldsymbol{\theta}_0$

- If only one additional "zero" constraint is considered $(\theta_i = 0)$, F_{W} becomes square of the Z statistic for θ_i (called C.R. in AMOS)

$$F_{\mathrm{W}} = \hat{\theta}_i^2 \Big/ \mathrm{avar}\left(\hat{\theta}_i\right)$$

- The LR, LM and W tests are asymptotically equivalent --- they're all about the same fit change, except for differently defined sampling error

- Which of the LM or the Wald test fits better into the logic of null hypothesis significance testing? Does it really matter? See Fig 7.5, p. 295

Respecification

- First of all, don't forget that SEM better serves confirmatory research questions --- implying that you should start with a reasonably "correct" model

- Consider different hierarchy of model structure in respecification, instead of only looking at F_{LM} or F_{W}:
  - model "configuration"
  - parameters near the observed variables vs. far

- Any respecification based on F_{LM} or F_{W} should be substantively justifiable; otherwise, it could be nothing but capitalizing on errors

- Also, researchers should try to exhaust all substantively interpretable models even when a satisfactory fit is attained

- Limitations of exploratory respecification, based on a sample:
  - LM and Wald tests are dependent on the fit model (importantly on where you start)
  - Like stepwise regression, there is order effects
  - Some alarming evidence from simulation studies against exploratory use of LM and Wald tests (Herbing & Costner, 1985; MacCallum, 1986)
- The exploratory use is most beneficial when
  - The initial model is not so much misspecified
  - Large N and
  - Resepecification is considered only for a particular part of the model --- i.e., sure about the other constraints or free parameters

- Significant chi-square change doesn’t necessarily mean a substantively meaningful parameter change --- N matters

- LOOK at residuals --- can suggest where the problems are, but it may not be so obvious why and how they happen

- Piecewise model fitting --- breaking the problem into smaller and easy pieces, particularly for a complicated model

Factor scores

- Estimation of factor scores is inherently indeterminant, regardless of EFA or CFA
- Essentially because too many unknowns (n common factors $+$ q error terms) compared to knowns (q indicators)
- The most common approach is regression in an unusual direction (predicting the latent with the observed); the resulting regression weights called “factor-score weights” --- different from loadings which are sometimes called “factor weights”
- Since any estimate of FS is fallible, replacing measurement models with FS estimates (treating them as observed variables) does not provide consistent estimates of path coefficients

Mean structure

- Modeling so far excluded mean structure, which is usual in modeling covariance structure (for a single group)

- Cases when to consider the mean structure:

  - Comparison of heterogeneous groups in factor means

  - Multilevel modeling --- means in nested groups interferes with covariance structure unless properly addressed

  - Comparison of item (or subscale) difficulties

  - When missing data need be treated along with the analysis --- most SEM programs offer missing imputation by model expectation assuming “missing at random”

- Mean structure included as an additional part of the model without affecting the covariance structure:

$$\mathbf{x} = \boldsymbol{\nu} + \boldsymbol{\Lambda}\boldsymbol{\xi} + \boldsymbol{\delta}, \quad E(\boldsymbol{\xi}) = \boldsymbol{\kappa}, \quad E(\boldsymbol{\delta}) = \mathbf{0}$$

$$E(\mathbf{x}) = \boldsymbol{\nu} + \boldsymbol{\Lambda}\boldsymbol{\kappa}, \quad E(\tilde{\mathbf{x}}\tilde{\mathbf{x}}') = \boldsymbol{\Lambda}\boldsymbol{\Phi}\boldsymbol{\Lambda}' + \boldsymbol{\Theta}$$

- Common scaling convention --- 0-intercept and 1-loading for one indicator per factor (e.g., 3 indicators for each of 2 factors):

$$\boldsymbol{\nu} = \begin{bmatrix} 0 \\ \nu_2 \\ \nu_3 \\ 0 \\ \nu_5 \\ \nu_6 \end{bmatrix}, \qquad \boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ \lambda_{21} & 0 \\ \lambda_{31} & 0 \\ 0 & 1 \\ 0 & \lambda_{52} \\ 0 & \lambda_{62} \end{bmatrix}$$

Higher-order FA

$$\mathbf{x} = \mathbf{\Lambda}_{(1)}\boldsymbol{\xi}_{(1)} + \boldsymbol{\delta}, \qquad \mathbf{\Sigma} = \mathbf{\Lambda}_{(1)}\mathbf{\Phi}_{(1)}\mathbf{\Lambda}'_{(1)} + \mathbf{\Theta}$$

$$\boldsymbol{\xi}_{(1)} = \mathbf{\Lambda}_{(2)}\boldsymbol{\xi}_{(2)} + \boldsymbol{\zeta}_{(1)}, \quad \mathbf{\Phi}_{(1)} = \mathbf{\Lambda}_{(2)}\mathbf{\Phi}_{(2)}\mathbf{\Lambda}'_{(2)} + \mathbf{\Psi}_{(1)}$$

$$\vdots \qquad\qquad\qquad \vdots$$

- Higher-order factors account for covariance between lower-order factors, not between lower-order error terms (e.g., g-intelligence underlying specific kinds of intelligence)
- Path modeling of latent variables explains covariances between (1^{st} order) factors through particularly specified directional paths whereas higher-order FA explains them by existence of higher-order factors (as common causes)