Chart to facilitate the calculation of partial coefficients of correlation and regression equations,

Kelley, Truman Lee, 1884-
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SPECIAL MONOGRAPH NO. 1

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BY
TRUMAN L. KELLEY 1884-
Professor of Education

STANFORD UNIVERSITY, CALIFORNIA
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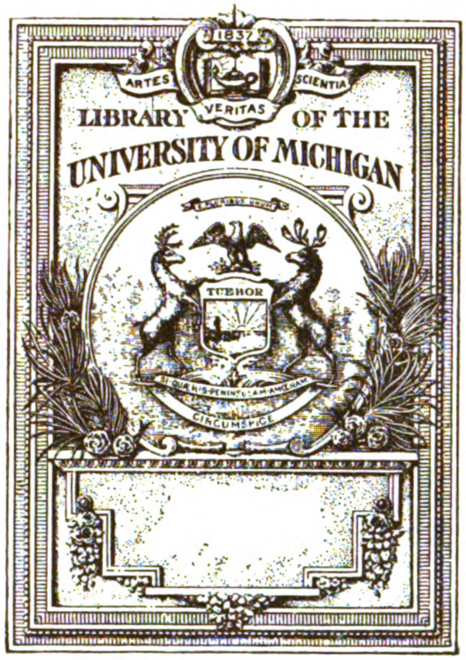
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PREFACE

In 1916 I published, as Bulletin 27 of the University of Texas, "Tables: To Facilitate the Calculation of Partial Coefficients of Correlation and Regression Equations." The present work, in terms of its purpose but not of its method, constitutes a second edition of that work. The preface to the "Tables" is as follows:

> The regression equation method has been so laborious, as well as involving such accuracy in and knowledge of statistical method, that it has not been used in many studies in which it alone could evaluate the data in such a manner as to answer the questions involved. It is hoped that the tables here presented have so materially decreased the labor of calculation that the method will be used extensively. If this has been accomplished a second edition will be demanded, and if such is called for two important improvements may be expected: first, that the tables be carried at least two decimal places further and, second, that entries for at least one of the variables be for every .001 instead of as at present for every .01. The shortcomings mentioned are well recognized. The author would be glad to hear of any others discovered by users.

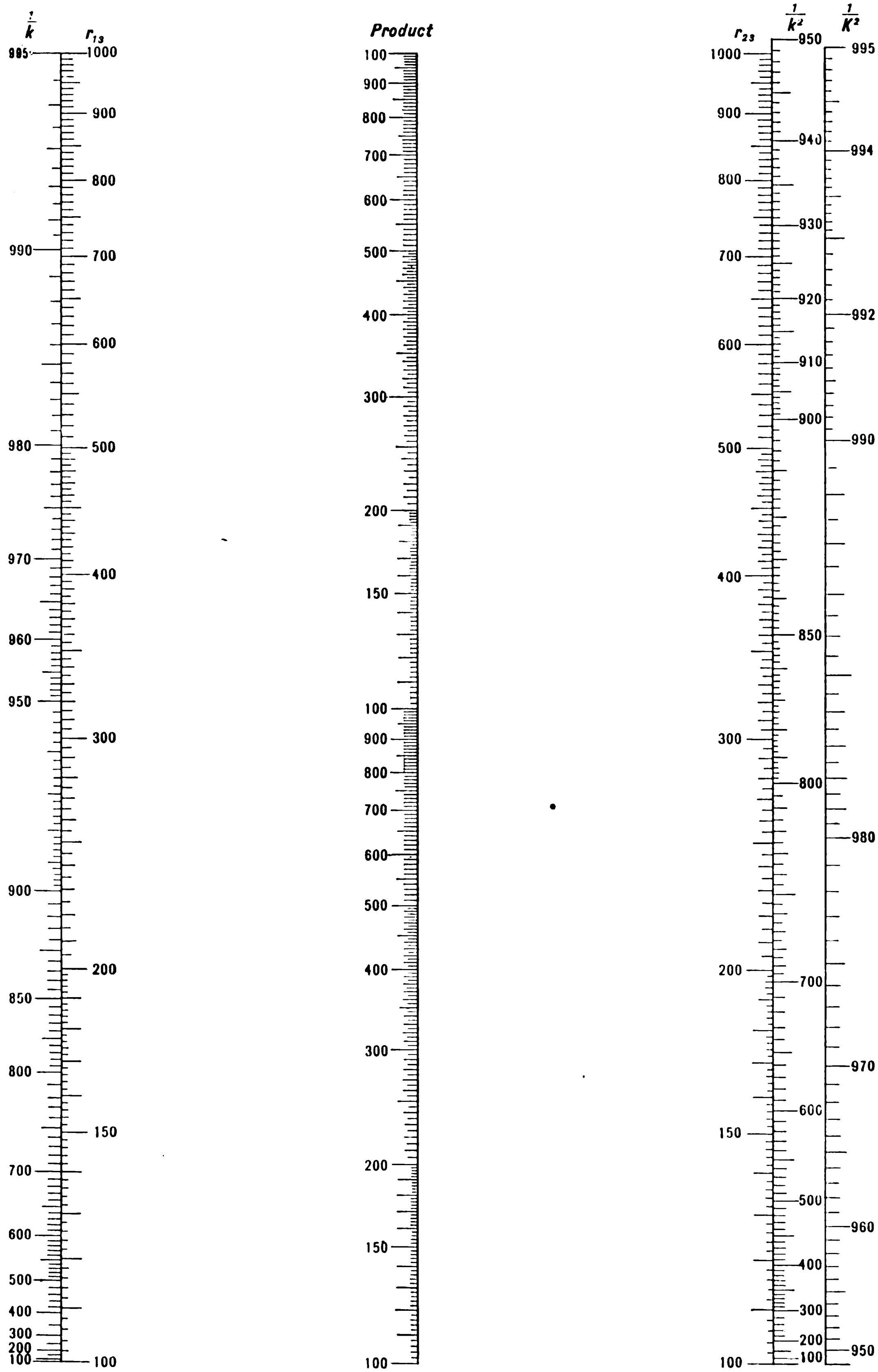
That the "Tables" met a real need has been proven by the early exhaustion of the edition and by the insistent demands for republication. However, certain shortcomings have made it undesirable to republish. These are, first, the labor involved in double interpolation and, second, the presence of cumulative inaccuracies in case the work is not carried to a sufficient number of decimal places and in case several variables are involved. These defects would only in part be remedied by extending the tables as suggested in the first preface. A final and decisive reason for not reprinting has been the discovery of the adaptability of the alignment chart to regression equation calculation.

I have tried the following methods of calculation of regression coefficients: (*a*) slide rule; (*b*) logarithmic; (*c*) by use of my Tables; (*d*) by use of the Chart here given; (*e*) by use of determinants; and (*f*) by use of successive approximations which more and more nearly approach the true values. (Method (*f*) will be described in detail in a forthcoming treatise.) I have found that the method which proves the best, depends upon the number of variables involved and in certain cases upon the degree of accuracy required. In general, if there are three or four variables, the Chart method will serve to advantage (also for five or six variables, if an inaccuracy in the regression coefficients in the neighborhood of .03 is not prohibitive); if five or six variables, the determinant method (utilizing a calculating machine rather than logarithms) will be most accurate and satisfactory; and if over six variables, the convergent series method of approximation will be the most expeditious and will result, by carrying the work the required number of steps, in any desired reliability.

The chart herewith described is accordingly especially recommended for use in problems involving three or four variables.

A new notation and procedure for the calculation of partial regression coefficients are used which materially simplify both the theoretical treatment and the arithmetical calculation. The notation includes three new symbols, z, k, and β, and the procedure permits the calculation of the regression coefficient of a given order by means of regression coefficients of an order one lower, thus entirely obviating the calculation of partial correlation coefficients and partial standard deviations.

Alignment Chart of Correlation Functions



SECTION I. THE MEANING OF PARTIAL CORRELATION

It is assumed that the meaning of the Pearson product-moment coefficient of correlation is well known to the reader and that the following symbols require no further exposition:

X_1 is the magnitude of the first, the dependent, variable.

X_2 is the magnitude of the second variable.

X_3 is the magnitude of the third variable, etc.

x_1 is the magnitude of the first variable expressed as a deviation from its own mean.

x_2, x_3, etc., have similar meanings for the second, third, etc., variables.

σ_1 is the standard deviation of the x_1's, σ_2 of the x_2's, etc.

r_{12} is the Pearson product-moment coefficient of correlation between variables 1 and 2; accordingly

$$r_{12} = \frac{\Sigma x_1 x_2}{N\sigma_1\sigma_2}$$

r's with other subscripts have similar meanings.

b_{12} is the regression coefficient of variable 1 on 2. b's with other subscripts have similar meanings.

The magnitude $b_{12}\,x_2$ is an estimated x_1. Letting $\bar{x}_1$ stand for such estimates, we have $\bar{x}_1 = b_{12}x_2$. The regression coefficient b_{12} is such a value that if in the correlation table (the scatter diagram) a straight line with this slope is drawn (in other words the line represented by the equation $x_1 = b_{12}x_2$) the sum of the squares of the deviations of the observed x_1's from this line will be a minimum. The $\bar{x}_1$'s are thus the closest estimates of the x_1's which it is possible to obtain knowing the x_2's and assuming a rectilinear relationship. The regression coefficient should be used only after a study of the correlation table has justified this assumption. Should such study definitely show curvilinear relationship, a simple transmutation of the scores of one of the variables can frequently be made resulting in rectilinear regression. In the following treatment rectilinear regression of the original scores, or of transmuted scores, is taken for granted. In case of non-rectilinear regression, b_{12} still represents the slope of the best straight line regression which can be obtained, but the $\bar{x}_1$'s resulting would differ more from the observed x_1's than would $\bar{x}_1$'s determined by means of the appropriate curvilinear regression line.

Let $x_1 - \bar{x}_1 = x_{1\cdot 2}$. Then $x_{1\cdot 2}$ is the "error of estimate" or "residual." If N is the population, there are of course N such magnitudes.

Their mean equals zero and their standard deviation is given by the equation:

$$\sigma_{1\cdot 2} = \sigma_1\sqrt{1 - r_{12}^2}$$

$\sigma_{1\cdot 2}$ is the "standard error of estimate" and is smaller than would have been the case had any value other than b_{12} been used. This "best" value is given by the equation:

$$b_{12} = \frac{\Sigma x_1 x_2}{N\sigma_2^2}$$

If x_2 scores are estimated, knowing x_1 scores, the regression equation is:

$$\bar{x}_2 = b_{21}x_1\text{, in which } b_{21} = \frac{\Sigma x_1 x_2}{N\sigma_1^2}$$

The difference between x_2 and $\bar{x}_2$ is $x_{2\cdot 1}$ and the standard deviation of these residuals, $\sigma_{2\cdot 1}$, is the standard error of estimate of the x_2's.

If the standard deviations of the two variables are equal

$$b_{12} = b_{21} = r_{12}$$

a measure of ***mutual*** implication. It is desirable, whether standard deviations are equal or not, to have a measure of mutual implication, and the coefficient of correlation continues such, though, in this case,

$$b_{12} \neq r \neq b_{21}$$

The relation between r_{12} and b_{12} and b_{21} is simple. Let us express each variable in terms of its own standard deviation and call the new variables obtained "standard measures," z_1 and z_2.

$$z_1 = \frac{x_1}{\sigma_1} \qquad \text{also } \bar{z}_1 = \frac{\bar{x}_1}{\sigma_1}$$

$$z_2 = \frac{x_2}{\sigma_2}$$

Substituting z's for x's in the equation,

$$\bar{x}_1 = \frac{\Sigma x_1 x_2}{N\sigma_2^2}\, x_2$$

and remembering that σ_1 and σ_2 are constants so that $\Sigma z_1\sigma_1 z_2\sigma_2 = \sigma_1\sigma_2\Sigma z_1 z_2$ gives:

$$\bar{z}_1 = \frac{\Sigma z_1 z_2}{N}\, z_2$$

and by a similar derivation

$$\bar{z}_2 = \frac{\Sigma z_1 z_2}{N}\, z_1$$

The common regression coefficient obtained when variables are expressed in terms of their own standard deviations is the coefficient of correlation, the desired measure of mutual implication. Therefore as a matter of definiton:

$$r_{12} = \frac{\Sigma z_1 z_2}{N} = \frac{\Sigma x_1 x_2}{N\sigma_1\sigma_2}$$

and as an immediate consequence

$$r_{12} = \sqrt{b_{12}b_{21}}$$

The correlation between variables 1 and 2 could be designated either as r_{12} or r_{21}, but custom places the numerically smaller subscript first.

It will be found serviceable in the following treatment to think of the correlation coefficient as simply the regression coefficient which exists when variables are expressed in terms of their own standard deviations.

If a third variable, x_3, is involved and it is desirable to obtain as accurate estimates of the x_1's as possible, knowing the x_2's and x_3's, it is done by the equation:

$$\bar{\bar{x}}_1 = b_{12\cdot3}\, x_2 + b_{13\cdot2}\, x_3$$

(The two dashes over the x_1 distinguish it from the preceding $\bar{x}_1$.) In general $b_{12\cdot3}$ will not be identical with b_{12}, for it is now necessary to weight x_2 in such a manner that *when combined with an appropriately weighted* x_3 the two together will yield an estimated x_1, $\bar{\bar{x}}_1$, which will correlate as highly as possible with x_1. In other words, the errors of estimate, or residuals, $x_1 - \bar{\bar{x}}_1 = x_{1.23}$, are to be as small as possible. More accurately stated, the standard error of estimate, $\sigma_{1\cdot23}$, is to be a minimum.

Logically it is obvious that if x_3 has any value whatever independent of x_2 in indicating x_1 scores then $\sigma_{1\cdot23} < \sigma_{1\cdot2}$. Also $\sigma_{1\cdot23} < \sigma_{1\cdot3}$. It may readily be proven by calculus that if the standard error of estimate is a minimum:

$$b_{12\cdot3} = \frac{r_{12} - r_{13}r_{23}}{1 - r_{23}^2} \times \frac{\sigma_1}{\sigma_2} \text{ and } b_{13\cdot2} = \frac{r_{13} - r_{12}r_{23}}{1 - r_{23}^2} \times \frac{\sigma_1}{\sigma_3}$$

It will aid thinking to have a concept corresponding to that part of $b_{12\cdot3}$ which does not depend upon the standard deviations of the variables. Calling that part $\beta_{12\cdot3}$ we have:

$$b_{12\cdot3} = \beta_{12\cdot3}\,\frac{\sigma_1}{\sigma_2} \text{ and } b_{13\cdot2} = \beta_{13\cdot2}\,\frac{\sigma_1}{\sigma_3}$$

The regression equation may then be written either as

$$\bar{\bar{x}}_1 = b_{12\cdot3}\, x_2 + b_{13\cdot2}\, x_3 \quad \text{or} \quad \bar{\bar{z}}_1 = \beta_{12\cdot3}\, z_2 + \beta_{13\cdot2}\, z_3$$

The chart herewith given enables the ready calculation of the constants of the type $\beta_{12\cdot 3}$. From this point on, the general treatment will be in terms of z's, and to return to x's the simple substitution,

$$z_1 = \frac{x_1}{\sigma_1}$$

is all that is necessary. The numerical illustration given is in terms of x's.

The weights, or regression coefficients, $\beta_{12\cdot 3}$ and $\beta_{13\cdot 2}$, are such that z_2 is weighted according to its relationship with z_1 independent of whatever common relationship z_3 may have with both z_1 and z_2; and similarly z_3 is weighted in terms of its importance independent of z_2. That, in order to obtain the minimum standard error of estimate, the weightings of z_2 and z_3 must be according to their relationships with z_1 independent of each other, is probably a mere matter of logic which the keen philosopher can see directly and without elaborate exposition, but the writer has been convinced of this fact only as a result of mathematical analysis. A numerical illustration in which the steps parallel those involved in the general analysis will serve to make clear the concrete meaning of the partial coefficient of correlation and the regression coefficient. Three variables will be considered and, to recapitulate, the notation will be as follows:

$X_1 =$ gross score of variable 1

$M_1 =$ the mean X_1 score

$x_1 = X_1 - M_1$

$\sigma_1 =$ the standard deviation of the x_1's

$z_1 = \dfrac{x_1}{\sigma_1}$

Variables 2 and 3 are similarly expressed

$r_{12} =$ the correlation of variables 1 and 2

$b_{12} =$ the regression of x_1 on $x_2 = \beta_{12} \dfrac{\sigma_1}{\sigma_2}$

$\beta_{12} =$ the regression of z_1 on $z_2 = b_{12} \dfrac{\sigma_2}{\sigma_1}$

(In case of two variables $\beta_{12} = r_{12}$, but in the case of three variables $\beta_{12\cdot 3} \neq r_{12\cdot 3}$)

Let k_{12} be defined by the equation $r_{12}^2 + k_{12}^2 = 1$. k is thus a magnitude which measures lack of relationship between variables 1 and 2. It will be called a "coefficient of alienation" and will be found to be as useful in interpreting data as the coefficient of correlation.[1]

[1] See Kelley, Truman L., "Principles Underlying the Classification of Men." *Jour. of Applied Psych.*, **3**, 50-67, March, 1919.

$$\sigma_{1\cdot 2} = \sqrt{\frac{\Sigma x^2_{1\cdot 2}}{N}}\text{, in which } x_{1\cdot 2} = x_1 - \bar{x}_1 \; (\bar{x}_1 = b_{12}\, x_2)$$

$$\sigma_{1\cdot 23} = \sqrt{\frac{\Sigma x^2_{1\cdot 23}}{N}}\text{, in which } x_{1\cdot 23} = x_1 - \bar{\bar{x}}_1$$

$$(\bar{\bar{x}}_1 = b_{12\cdot 3}\, x_2 + b_{13\cdot 2}\, x_3)$$

Comparable meanings for σ's with other subscripts. With this notation we will consider the relationships in the three accompanying series, x_1, x_2, x_3.

	x_1	x_2	x_3	$\bar{x}_1$	$x_{1\cdot 3}$	$x_{2\cdot 3}$	$\bar{x}_{1\cdot 3}$	$x_{1\cdot 23}$	$\bar{\bar{x}}_1$
	—12	—2	—3	—9	—3	1	— .8	—2.2	—9.8
	—8	—2	—2	—6	—2	0	.0	—2.0	—6.0
	—4	—2	—2	—6	2	0	.0	2.0	—6.0
	4	—6	—1	—3	7	—5	4.0	3.0	1.0
	0	—4	—1	—3	3	—3	2.4	.6	— .6
	0	0	—1	—3	3	1	— .8	3.8	—3.8
	—8	0	0	0	—8	0	.0	—8.0	.0
	—4	2	0	0	—4	2	—1.6	—2.4	—1.6
	—4	2	0	0	—4	2	—1.6	—2.4	—1.6
	0	6	0	0	0	6	—4.8	4.8	—4.8
	8	—4	1	3	5	—5	4.0	1.0	7.0
	4	4	1	3	1	3	—2.4	3.4	.6
	4	0	1	3	1	—1	.8	.2	3.8
	12	0	2	6	6	—2	1.6	4.4	7.6
	0	2	2	6	—6	0	.0	—6.0	6.0
	8	4	3	9	—1	1	— .8	— .2	8.2
Sums	0	0	0	0	0	0	.0	.0	.0
Standard deviations.....	$\sqrt{\frac{640}{16}}$	$\sqrt{\frac{160}{16}}$	$\sqrt{\frac{40}{16}}$	$\sqrt{\frac{360}{16}}$	$\sqrt{\frac{280}{16}}$	$\sqrt{\frac{120}{16}}$	$\sqrt{\frac{76.8}{16}}$	$\sqrt{\frac{203.2}{16}}$	$\sqrt{\frac{436.8}{16}}$

For convenience the three series have been so chosen that the mean of each is zero, so that the measures recorded are x measures.

$$r_{12} = \frac{3}{40}\text{, } r_{13} = \frac{3}{4}\text{, and } r_{23} = \frac{1}{2}$$

We may use series 3 to estimate series 1 by the equation,

$$\bar{x}_1 = b_{13}\, x_3\text{, in which } b_{13} = r_{13}\frac{\sigma_1}{\sigma_3} = 3$$

The values recorded in column $\bar{x}_1$ are such values. These estimated values of x_1 differ from the actual x_1 values by the amounts shown in column $x_{1\cdot 3}$ (to be read "the errors of estimate of the x_1's when estimated from the x_3's," or "the residuals in the x_1's after estimation from the x_3's"). Thus far x_2 has not been used. We, of course, cannot use the x_3's to estimate these residuals, $x_{1\cdot 3}$'s, as the correlation between the $x_{1\cdot 3}$'s and the x_3's is of necessity zero. We must resort to an additional source of data, such as is series 2, to reduce the error of estimate still further. However, in so far as series 2 is related to series 3 it will not be of service. Instead, therefore, of correlating series 2 with the residuals, $x_{1\cdot 3}$, we will correlate that part of series 2 which is independent of series 3 with these residuals. $x_{2\cdot 3}$ is the part desired. These are the residuals in the x_2's when x_2's are estimated from x_3's. They are obtained in a manner comparable to that of the $x_{1\cdot 3}$'s, being given by the equation,

$$x_{2\cdot 3} = x_2 - b_{23}\, x_3 \text{, in which } b_{23} = r_{23}\,\frac{\sigma_2}{\sigma_3} = 1$$

These residuals, recorded in column $x_{2\cdot 3}$, may be used to estimate the residuals, $x_{1\cdot 3}$, thus leading to a closer approximation of the x_1 scores than is possible by utilizing series 3 only. The correlation between $x_{1\cdot 3}$ and $x_{2\cdot 3}$ is

$$r_{(1\cdot 3)(2\cdot 3)} = \frac{-96}{\sqrt{280}\,\sqrt{120}}$$

Accordingly the regression coefficient of the $x_{1\cdot 3}$'s upon the $x_{2\cdot 3}$'s is

$$\bar{x}_{1\cdot 3} = r_{(1\cdot 3)(2\cdot 3)}\,\frac{\sigma_{1\cdot 3}}{\sigma_{2\cdot 3}} = -.8$$

The correlation coefficient, $r_{(1\cdot 3)(2\cdot 3)}$, is usually expressed as $r_{12\cdot 3}$ and may be read "the correlation between those parts of x_1 and x_2 which are independent of x_3"; in other words, "the partial correlation between x_1 and x_2 when the relationship of x_3 is eliminated." Or again, "the correlation of x_1 and x_2 independent of x_3," or "the correlation between x_1 and x_2 when x_3 is constant." Using the regression coefficient just obtained, $-.8$, to estimate $x_{1\cdot 3}$ residuals from the $x_{2\cdot 3}$ residuals gives the measures in column $\bar{x}_{1\cdot 3}$. The differences between these and the $x_{1\cdot 3}$ residuals are recorded in column $x_{1\cdot 23}$ and are the errors of estimate or residuals when both x_2 and x_3 have been utilized to their fullest extent in estimating x_1 scores. The standard deviation of these final residuals is

$$\sigma_{1\cdot 23} = \sqrt{\frac{203.2}{16}}$$

This is the standard error of estimate, $\sigma_{1\cdot 23}$. The probable error of estimate $= .6745\ \sigma_{1\cdot 23} = 2.4037$. Accordingly if x_1 scores are estimated from x_2 and x_3 scores, the resulting estimate probably will be in error by as much as 2.4037.

Finally, if the estimate of the residual $x_{1\cdot 3}$, namely $\bar{x}_{1\cdot 3}$, be added to the estimate of x_1, namely $\bar{x}_1$, a magnitude $\bar{\bar{x}}_1$, is obtained which is the estimate of the x_1's obtained by utilizing both x_2 and x_3. $\bar{\bar{x}}_1$ is thus given by the equation:

$$\bar{\bar{x}}_1 = \bar{x}_1 + \bar{x}_{1\cdot 3} = r_{13}\frac{\sigma_1}{\sigma_3}x_3 + r_{12\cdot 3}\frac{\sigma_{1\cdot 3}}{\sigma_{2\cdot 3}}x_{2\cdot 3}$$

It can be proven that the correlation between x_1 and $\bar{\bar{x}}_1$, designated by $r_{1\cdot 23}$ and called the multiple correlation, is the maximum obtainable. The difference between x_1 and $\bar{\bar{x}}_1$ is of course $x_{1\cdot 23}$, already obtained in a slightly different manner. The equation just given for $\bar{\bar{x}}_1$ could be used to estimate x_1 scores knowing x_2 and x_3 scores, but it would necessitate the calculation of magnitudes $x_{2\cdot 3}$. Accordingly this equation has no practical utility, except as here given in illustrating how x_3's and x_2's may be fully utilized to estimate x_1's. Algebraic reduction (involving the expressing of partials in terms of totals, e. g., $x_{2\cdot 3} = x_2 - b_{23}\,x_3$, etc., for all factors involved in $\sigma_{1\cdot 3}$, $\sigma_{2\cdot 3}$ and $r_{12\cdot 3}$) yields:

$$\frac{\bar{\bar{x}}_1}{\sigma_1} = \frac{r_{12} - r_{13}r_{23}}{1 - r_{23}^2}\,\frac{x_2}{\sigma_2} + \frac{r_{13} - r_{12}r_{23}}{1 - r_{23}^2}\,\frac{x_3}{\sigma_3}$$

(equivalent to $\bar{\bar{z}}_1 = \beta_{12\cdot 3}\,z_2 + \beta_{13\cdot 2}\,z_3$).

The problem before us, then, is primarily to determine such regression coefficients as $\beta_{12\cdot 3}$. The chart here given was drawn up to facilitate such determinations. It, however, is equally serviceable in enabling the calculation of $r_{12\cdot 3}$, $\sqrt{1 - r_{12}^2}$ designated as k_{12}, and other correlation functions.

SECTION II. FORMULAS INVOLVED IN MULTIPLE CORRELATION.

The basic problem of multiple correlation is to estimate the value of one variable, knowing the values of several others. Provided relationships are rectilinear, or approximately so, this problem is solved by means of an equation:

$$\bar{X}_1 = b_{12\cdot 34\ldots n} X_2 + b_{13\cdot 24\ldots n} X_3 + \cdots + b_{1n\cdot 23\ldots n-1} X_n + c \qquad (1)$$

in which the b's and the c are constants, so chosen that when the variables $X_2, X_3, \cdots X_n$ are multiplied by the successive b's and added to c a final measure, $\bar{X}_1$, is obtained which is the most accurate estimate of X_1 possible of attainment.

If measures are expressed as deviations from their own means divided by their own standard deviations the above regression equation simplifies. Let

$$\bar{z}_1 = \frac{\bar{X}_1 - M_1}{\sigma_1},\ z_2 = \frac{X_2 - M_2}{\sigma_2},\ \text{etc.,}$$

in which the M's are successive means and the σ's successive standard deviations. There is thus a one-to-one relation between the z's and the X's. The regression equation connecting the z's is:

$$\bar{z}_1 = \beta_{12\cdot 34\ldots n} z_2 + \beta_{13\cdot 24\ldots n} z_3 + \cdots + \beta_{1n\cdot 23\ldots n-1} z_n \qquad (2)$$

in which the b's and β's are connected by the relationships:

$$b_{12\cdot 34\ldots n} = \beta_{12\cdot 34\ldots n} \frac{\sigma_1}{\sigma_2}$$

$$b_{13\cdot 24\ldots n} = \beta_{13\cdot 24\ldots n} \frac{\sigma_1}{\sigma_3},\ \text{etc.} \qquad (3)$$

Also:

$$c = M_1 - b_{12\cdot 34\ldots n} M_2 - b_{13\cdot 24\ldots n} M_3 - \cdots - b_{1n\cdot 23\ldots n-1} M_n \qquad (4)$$

The β constants involved in equation (2) are the regression coefficients derived by aid of the chart. Knowing equation (2) and having relationships (3) and (4) it is but a step to secure equation (1) which is the most serviceable form of the regression equation for actual use.

The probable error of the estimated X_1's, i. e., of the $\bar{X}_1$'s, is .6745 $\times$ their standard error, or .6745 $\times$ their standard deviation, $\sigma_{1\cdot 23\ldots n}$.

Coefficients of alienation, k, may be defined by the type equation:

$$k_{12\cdot 34\ldots n} = \sqrt{1 - r^2_{12\cdot 34\ldots n}} \qquad (5)$$

In the case of two variables the following continued equality may be written:

$$k_{12} = k_{1\cdot 2} = \sqrt{1 - r_{12}^2} = \sqrt{1 - r_{1\cdot 2}^2} \tag{5a}$$

[for $k_{1\cdot 2}$ and $r_{1\cdot 2}$ see (6) and (7)]

and in general, $k^2 + r^2 = 1$, where the subscript of k is identical with the subscript of r.

It may be shown that:

$$\sigma_{1\cdot 23 \ldots n} = \sigma_1 \; k_{1\cdot 23 \ldots n}, \tag{6}$$

in which

$$k_{1\cdot 23 \ldots n} = k_{12} \; k_{13\cdot 2} \; k_{14\cdot 23} \; k_{15\cdot 234} \cdots k_{1n\cdot 23 \ldots n-1} \tag{7}$$

k_{12} may be read, "the coefficient of alienation between 1 and 2."

$$\sigma_{1\cdot 2} = \sigma_1 \cdot k_{12}$$

Accordingly k_{12} is a measure of variability, or freedom of variable 1 from variable 2 which is independent of the standard deviations of the variables. If 1 is the dependent variable and 2 the independent it will be convenient, in place of k_{12}, to use the notation $k_{1\cdot 2}$, which may be read, "the freedom of 1 from 2." Similarly $k_{1\cdot 23}$ is the freedom of 1 from 2 and 3, or "the variability in 1 unaccounted for by 2 and 3." With this explanation the meaning of such symbols as $k_{2\cdot 13}$; $k_{1\cdot 23 \ldots n}$, $k_{3\cdot 124 \ldots n}$, etc., will be obvious.

Further, if we write,

$$k_{1\cdot 23}^2 + r_{1\cdot 23}^2 = 1 \quad (\text{in general: } k_{1\cdot 23 \ldots n}^2 + r_{1\cdot 23 \ldots n}^2 = 1) \tag{8}$$

$r_{1\cdot 23}$ is the correlation between 1, and 2 and 3 when combined by the regression equation. The symbol here given, $r_{1\cdot 23}$, is an extension of the notation introduced by Yule [1] in the symbol $\sigma_{1\cdot 23}$. The relationship is

$$\therefore \; \sigma_{1\cdot 23} = \sigma_1 \; k_{1\cdot 23} = \sigma_1 \sqrt{1 - r_{1\cdot 23}^2}.$$

The standard error of estimate, $\sigma_{1\cdot 23 \ldots n}$, is given in terms of k in equation (6). k with a single primary subscript, e. g., $k_{1\cdot 23 \ldots n}$, is given in terms of k's with two primary subscripts in equation (7). k's with two

[1] Yule, G. Udny, "An Introduction to the Theory of Statistics." Lippincott, 1912. The symbol $r_{1\cdot 23 \ldots n}$ differs from the symbol $R_{1(23 \ldots n)}$ which Yule gives, but as capital R has earlier and with much pertinence been used by Pearson and others to designate certain correlation determinants it is undesirable to use it as a multiple correlation coefficient. Pearson has used various symbols including R_n and ϱ. Furthermore, the relationship between $\sigma_{1\cdot 23 \ldots n}$, $k_{1\cdot 23 \ldots n}$ and $r_{1\cdot 23 \ldots n}$ is an argument in favor of both symbols, $k_{1\cdot 23 \ldots n}$ and $r_{1\cdot 23 \ldots n}$.

primary subscripts are given in terms of r's in equations (5) and (5a) and in terms of β's in the following equations:

$$k_{12} = \sqrt{1 - \beta_{12}\beta_{21}}$$

(When there are no secondary subscripts $\beta_{12} = \beta_{21} = r_{12}$)

$$k_{12\cdot3} = \sqrt{1 - \beta_{12\cdot3}\beta_{21\cdot3}}, \text{ and in general}$$

$$k_{12\cdot34\cdots n} = \sqrt{1 - \beta_{12\cdot34\cdots n}\,\beta_{21\cdot34\cdots n}} \tag{9}$$

Partial coefficients of correlation may be found from partial regression coefficients:

$$r_{12\cdot34\cdots n} = \sqrt{\beta_{12\cdot34\cdots n}\,\beta_{21\cdot34\cdots n}} \tag{10}$$

These two β's may appropriately be called "conjugate regression coefficients." The elements entering into them are the same and they are involved in a reciprocal manner. Whatever the sign of $\beta_{12\cdot34\cdots n}$, the partial regression of 1 upon 2, it is likewise the sign of $\beta_{21\cdot34\cdots n}$, the partial regression of 2 upon 1. This is the sign to be attached to $r_{12\cdot34\cdots n}$.

The foregoing formulas show that every correlation coefficient can be found if the β's are known.

The fundamental formula for expressing a β of a given order in terms of β's of lower order is:

$$\beta_{12\cdot34\cdots n} = \frac{\beta_{12\cdot45\cdots n} - \beta_{13\cdot45\cdots n}\,\beta_{32\cdot45\cdots n}}{1 - \beta_{23\cdot45\cdots n}\,\beta_{32\cdot45\cdots n}} \tag{11}$$

Following Yule (who, however, deals with b's instead of β's) $\beta_{12\cdot34\cdots n}$ may be called a regression coefficient of the $n - 2$ order, $\beta_{12\cdot45\cdots n}$ one of the $n - 3$ order, etc.—the order being determined by the number of secondary subscripts. β_{12} is accordingly a regression coefficient of zero order. The order of the secondary subscripts is immaterial, but the order of the primary subscripts is definite. In equation (11) one (called the unique secondary subscript) and only one of the secondary subscripts appearing in the β in the left-hand member has disappeared from the secondary subscripts in the β's in the right-hand member. Since all but one of the secondary subscripts appear as secondary subscripts in both members the general principle may be illustrated by a β of the second order:

$$\beta_{12\cdot34} = \frac{\beta_{12\cdot4} - \beta_{13\cdot4}\beta_{32\cdot4}}{1 - \beta_{23\cdot4}\beta_{32\cdot4}}$$

The first primary subscript in the left-hand member term becomes the first primary subscript in the first and second β's in the numerator of the right-hand member. The second primary subscript in the left-hand member term becomes the second primary subscript in the first and third β's in the numerator. The remaining two primary subscipts of the numerator β's are identical and are the unique secondary subscript. The denominator β's are the third numerator β and its conjugate.

From these general directions it is obvious that there are as many different ways for expressing a regression coefficient of a given order as the order of the coefficient. $\beta_{12\cdot34\cdots n}$ may be expressed in $n-2$ ways. Equation (11) is one such, and the following is another:

$$\beta_{12\cdot34\cdots n} = \frac{\beta_{12\cdot35\cdots n} - \beta_{14\cdot35\cdots n}\,\beta_{42\cdot35\cdots n}}{1 - \beta_{24\cdot35\cdots n}\,\beta_{42\cdot35\cdots n}}$$

In practice it is desirable to calculate in at least two ways as a check.

Formula (11) simplifies in case a partial regression coefficient of the first order is being calculated:

$$\beta_{12\cdot3} = \frac{\beta_{12} - \beta_{13}\beta_{32}}{1 - \beta_{23}\beta_{32}} = \frac{r_{12} - r_{13}r_{23}}{1 - r_{23}^2} \tag{11a}$$

By repeated use of formulas (11) and (11a) in calculating regression coefficients of a given order from those of an order one less, every regression coefficient may be obtained. By formulas (6), (7), and (9) the standard error of estimate, or the standard deviation of the residuals, may be found. Finally, by equation (8), which may be written in the form:

$$r_{1\cdot23\cdots n} = \sqrt{1 - k_{1\cdot23\cdots n}^2} \tag{8}$$

the multiple coefficient of correlation between one variable and any number of others may be found.

The accompanying table indicates the partial regression coefficients of the first and second orders which will be needed to complete the solution of a four-variable problem. The outline provides for the calculation of each second order partial in two ways. The first time a coefficient appears in the table it is designated by a number in parentheses, or, in case it and its conjugate are both required, by a letter in parentheses. The lack of a number or letter before a coefficient indicates that it has appeared earlier in the table.

NEEDED	MAY BE OBTAINED FROM			MAY ALSO BE OBTAINED FROM		
$\beta_{12\cdot34}$	(1)$\beta_{12\cdot4}$	(2)$\beta_{13\cdot4}$	(*a*)$\beta_{32\cdot4}$	(3)$\beta_{12\cdot3}$	(4)$\beta_{14\cdot3}$	(*b*)$\beta_{42\cdot3}$
$\beta_{13\cdot24}$	$\beta_{13\cdot4}$	$\beta_{12\cdot4}$	$\beta_{23\cdot4}$	(5)$\beta_{13\cdot2}$	(6)$\beta_{14\cdot2}$	(*c*)$\beta_{43\cdot2}$
$\beta_{14\cdot23}$	$\beta_{14\cdot3}$	$\beta_{12\cdot3}$	$\beta_{24\cdot3}$	$\beta_{14\cdot2}$	$\beta_{13\cdot2}$	$\beta_{34\cdot2}$

In addition to the preceding which suffice to obtain the regression coefficients, the following conjugates of coefficients already obtained will enable the calculation in two ways of the multiple alienation coefficient, $k_{1\cdot234}$, of the multiple correlation coefficient, $r_{1\cdot234}$, and of the standard error of estimate, $\sigma_{1\cdot234}$.

$\beta_{21\cdot34}$	$\beta_{21\cdot4}$	$\beta_{23\cdot4}$	$\beta_{31\cdot4}$	$\beta_{21\cdot3}$	$\beta_{24\cdot3}$	$\beta_{41\cdot3}$

$k^2_{1\cdot234}$ is calculated before $r_{1\cdot234}$ or $\sigma_{1\cdot234}$.

$$k^2_{1\cdot234} = (1 - \beta_{21\cdot34}\beta_{12\cdot34})(1 - \beta_{31\cdot4}\beta_{13\cdot4})(1 - \beta_{41}\beta_{14})$$

Also,

$$k^2_{1\cdot234} = (1 - \beta_{21\cdot34}\beta_{12\cdot34})(1 - \beta_{41\cdot3}\beta_{14\cdot3})(1 - \beta_{31}\beta_{13})$$

In the case of five variables the following outline may be followed:

NEEDED	MAY BE OBTAINED FROM			MAY ALSO BE OBTAINED FROM		
$(13)\beta_{12\cdot45}$	$(1)\beta_{12\cdot5}$	$(2)\beta_{14\cdot5}$	$(a)\beta_{42\cdot5}$	$(3)\beta_{12\cdot4}$	$(4)\beta_{15\cdot4}$	$(b)\beta_{52\cdot4}$
$(14)\beta_{13\cdot45}$	$(5)\beta_{13\cdot5}$	$\beta_{14\cdot5}$	$(c)\beta_{43\cdot5}$	$(6)\beta_{13\cdot4}$	$\beta_{15\cdot4}$	$(d)\beta_{53\cdot4}$
$(15)\beta_{14\cdot23}$	$(7)\beta_{14\cdot2}$	$(8)\beta_{13\cdot2}$	$(e)\beta_{34\cdot2}$	$(9)\beta_{14\cdot3}$	$(10)\beta_{12\cdot3}$	$(f)\beta_{24\cdot3}$
$(16)\beta_{15\cdot23}$	$(11)\beta_{15\cdot2}$	$\beta_{13\cdot2}$	$(g)\beta_{35\cdot2}$	$(12)\beta_{15\cdot3}$	$\beta_{12\cdot3}$	$(h)\beta_{25\cdot3}$
$(17)\beta_{12\cdot34}$	$\beta_{12\cdot4}$	$\beta_{13\cdot4}$	$(i)\beta_{32\cdot4}$	$\beta_{12\cdot3}$	$\beta_{14\cdot3}$	$\beta_{42\cdot3}$
$(18)\beta_{15\cdot34}$	$\beta_{15\cdot4}$	$\beta_{13\cdot4}$	$\beta_{35\cdot4}$	$\beta_{15\cdot3}$	$\beta_{14\cdot3}$	$(j)\beta_{45\cdot3}$
$(19)\beta_{13\cdot25}$	$\beta_{13\cdot5}$	$\beta_{12\cdot5}$	$(k)\beta_{23\cdot5}$	$\beta_{13\cdot2}$	$\beta_{15\cdot2}$	$\beta_{53\cdot2}$
$(20)\beta_{14\cdot25}$	$\beta_{14\cdot5}$	$\beta_{12\cdot5}$	$\beta_{24\cdot5}$	$\beta_{14\cdot2}$	$\beta_{15\cdot2}$	$(l)\beta_{54\cdot2}$
$(m)\beta_{23\cdot45}$	$\beta_{23\cdot5}$	$\beta_{24\cdot5}$	$\beta_{43\cdot5}$	$\beta_{23\cdot4}$	$\beta_{25\cdot4}$	$\beta_{53\cdot4}$
$(M)\beta_{32\cdot45}$	$\beta_{32\cdot5}$	$\beta_{34\cdot5}$	$\beta_{42\cdot5}$	$\beta_{32\cdot4}$	$\beta_{35\cdot4}$	$\beta_{52\cdot4}$
$(n)\beta_{45\cdot23}$	$\beta_{45\cdot2}$	$\beta_{43\cdot2}$	$\beta_{35\cdot2}$	$\beta_{45\cdot3}$	$\beta_{42\cdot3}$	$\beta_{25\cdot3}$
$(N)\beta_{54\cdot23}$	$\beta_{54\cdot2}$	$\beta_{53\cdot2}$	$\beta_{34\cdot2}$	$\beta_{54\cdot3}$	$\beta_{52\cdot3}$	$\beta_{24\cdot3}$
$(o)\beta_{25\cdot34}$	$\beta_{25\cdot4}$	$\beta_{23\cdot4}$	$\beta_{35\cdot4}$	$\beta_{25\cdot3}$	$\beta_{24\cdot3}$	$\beta_{45\cdot3}$
$(O)\beta_{52\cdot34}$	$\beta_{52\cdot4}$	$\beta_{53\cdot4}$	$\beta_{32\cdot4}$	$\beta_{52\cdot3}$	$\beta_{54\cdot3}$	$\beta_{42\cdot3}$
$(p)\beta_{34\cdot25}$	$\beta_{34\cdot5}$	$\beta_{32\cdot5}$	$\beta_{24\cdot5}$	$\beta_{34\cdot2}$	$\beta_{35\cdot2}$	$\beta_{54\cdot2}$
$(P)\beta_{43\cdot25}$	$\beta_{43\cdot5}$	$\beta_{42\cdot5}$	$\beta_{23\cdot5}$	$\beta_{43\cdot2}$	$\beta_{45\cdot2}$	$\beta_{53\cdot2}$
$\beta_{12\cdot345}$	$(13)\beta_{12\cdot45}$	$(14)\beta_{13\cdot45}$	$(M)\beta_{32\cdot45}$	$(17)\beta_{12\cdot34}$	$(18)\beta_{15\cdot34}$	$(O)\beta_{52\cdot34}$
$\beta_{13\cdot245}$	$\beta_{13\cdot45}$	$\beta_{12\cdot45}$	$(m)\beta_{23\cdot45}$	$(19)\beta_{13\cdot25}$	$(20)\beta_{14\cdot25}$	$(P)\beta_{43\cdot25}$
$\beta_{14\cdot235}$	$(15)\beta_{14\cdot23}$	$(16)\beta_{15\cdot23}$	$(N)\beta_{54\cdot23}$	$\beta_{14\cdot25}$	$\beta_{13\cdot25}$	$(p)\beta_{34\cdot25}$
$\beta_{15\cdot234}$	$\beta_{15\cdot23}$	$\beta_{14\cdot23}$	$(n)\beta_{45\cdot23}$	$\beta_{15\cdot34}$	$\beta_{12\cdot34}$	$(o)\beta_{25\cdot34}$

In addition to the preceding, the following conjugates enable the calculation in two ways of $k_{1\cdot2345}$, $r_{1\cdot2345}$ and $\sigma_{1\cdot2345}$.

$(13a)\beta_{21\cdot45}$	$(3a)\beta_{21\cdot4}$	$\beta_{25\cdot4}$	$(4a)\beta_{51\cdot4}$			
$(14a)\beta_{31\cdot45}$	$(6a)\beta_{31\cdot4}$	$\beta_{35\cdot4}$	$\beta_{51\cdot4}$			
$(17a)\beta_{21\cdot34}$	$(10a)\beta_{21\cdot3}$	$\beta_{24\cdot3}$	$(9a)\beta_{41\cdot3}$			
$(18a)\beta_{51\cdot34}$	$(12a)\beta_{51\cdot3}$	$\beta_{54\cdot3}$	$\beta_{41\cdot3}$			
$\beta_{21\cdot345}$	$\beta_{21\cdot45}$	$\beta_{23\cdot45}$	$\beta_{31\cdot45}$	$\beta_{21\cdot34}$	$\beta_{25\cdot34}$	$\beta_{51\cdot34}$

$$k^2_{1\cdot2345} = (1 - \beta_{21\cdot345}\beta_{12\cdot345})(1 - \beta_{31\cdot45}\beta_{13\cdot45})(1 - \beta_{41\cdot5}\beta_{14\cdot5})(1 - \beta_{15}\beta_{51})$$

Also,

$$k^2_{1\cdot2345} = (1 - \beta_{21\cdot345}\beta_{12\cdot345})(1 - \beta_{51\cdot34}\beta_{15\cdot34})(1 - \beta_{31\cdot4}\beta_{13\cdot4})(1 - \beta_{41}\beta_{14})$$

SECTION III. THE USE OF THE CHART.

The directions here given apply to the small chart in this monograph and also to a large twenty-inch chart, which may be secured separately and which gives results of approximately the same degree of accuracy as a twenty-inch slide rule.

The scales for r_{13} and r_{23} are graduated according to the logarithms of numbers from 10 to 100, and the product scale is so graduated as to indicate the products of any two numbers on scales r_{13} and r_{23} when connected by a straight line. Accordingly all products and quotients, including squares and square roots, may be obtained. In all these operations the simplest way to keep track of the decimal point is to roughly carry the operation through in one's head and then place the point where it belongs.

Scale $1/k$ is graduated according to the logarithms of $1/\sqrt{1-r^2}$ and scale $1/k^2$ according to the logarithms of $1/1-r^2$. Scale $1/K^2$ is a continuation of scale $1/k^2$. When values on scale $1/K^2$ are used, place a straight edge through this value and parallel to the base line [as explained in example (c)] and locate a point on scale $1/k^2$. Then continue the calculation using the point so located on scale $1/k^2$ in lieu of the point on scale $1/K^2$.

The following magnitudes are needed in multiple correlation work:

(a) Products, such as $r_{13}r_{23}$

(b) Quotients, such as $\frac{\sigma_1}{\sigma_2}$

(c) Square roots, such as $\sqrt{\beta_{12\cdot3}\beta_{21\cdot3}}$

(d) Factors $\frac{1}{k_{13}}$ $(=\frac{1}{\sqrt{1-r_{13}^2}})$ which enter in partial coefficients of correlation

(e) Coefficients of alienation, such as k_{13} $(=\sqrt{1-r_{13}^2})$

(f) Factors $\frac{1}{k_{23}^2}$ $(=\frac{1}{1-r_{23}^2})$ which enter into regression coefficients

(g) Squares of coefficients of alienation, such as k_{23}^2 $(=1-r_{23}^2)$

(h) Partial regression coefficients, such as $\beta_{12\cdot3}$ $(=\frac{r_{12}-r_{13}r_{23}}{k_{23}^2})$

(i) Partial correlation coefficients, such as

$$r_{12\cdot3}\ (=\frac{r_{12}-r_{13}r_{23}}{k_{13}k_{23}}=\sqrt{\beta_{12\cdot3}\beta_{21\cdot3}}=\sqrt{b_{12\cdot3}b_{21\cdot3}})$$

(*j*) Partial regression coefficients involving four variables

$$\beta_{12\cdot34}\ \left(= \frac{\beta_{12\cdot4} - \beta_{13\cdot4}\beta_{32\cdot4}}{k^2_{23\cdot4}} = \frac{\beta_{12\cdot3} - \beta_{14\cdot3}\beta_{42\cdot3}}{k^2_{24\cdot3}}\right)$$

Since $k^2_{23\cdot4} = 1 - \beta_{23\cdot4}\beta_{32\cdot4}$, and since the calculation which leads to $\beta_{23\cdot4}$ is changed in but one simple respect to obtain $\beta_{32\cdot4}$ it is convenient to write:

$$\beta_{12\cdot34} = \frac{\beta_{12\cdot4} - \beta_{13\cdot4}\beta_{32\cdot4}}{1 - \beta_{23\cdot4}\beta_{32\cdot4}}$$

(*k*) Partial regression coefficients involving more than four variables

$$\beta_{12\cdot34\ldots n} = \frac{\beta_{12\cdot4\ldots n} - \beta_{13\cdot4\ldots n}\,\beta_{32\cdot4\ldots n}}{1 - \beta_{23\cdot4\ldots n}\,\beta_{32\cdot4\ldots n}}$$

The same procedure as in (*j*) is followed, but in this case the calculation which leads to $\beta_{23\cdot4\ldots n}$ does not, by one simple change, lead to $\beta_{32\cdot4\ldots n}$.

Examples:

(*a*) $.2 \times .4$ Place a straight edge on 20, scale r_{13}, and upon 40, scale r_{23}, and read the product, .08, on the product scale.

(*b*) $\frac{2}{.4}$ Place a straight edge upon 20, product scale, and upon 40, scale r_{23}, and read the quotient, 5.0, on scale r_{13}.

(*c*) $\sqrt{.25}$ Place a straight edge on 25, product scale, and parallel to the base line of the chart (this can be done by rotating the straight edge until the readings on scales r_{13} and r_{23} are identical) and read the square root, .50, on either scale r_{13} or r_{23}.

(*d*) $\frac{1}{\sqrt{1 - .60^2}}$ Find 60 on scale $1/k$ and read the answer, 1.25, from the same point on scale r_{13}.

(*e*) $\sqrt{1 - .60^2}$ Place a straight edge through 60, scale $1/k$, and 100, product scale, and read the answer, .80, on scale r_{23}.

(*f*) $\frac{1}{1 - .60^2}$ Find 60 on scale $1/k^2$ and read the answer, 1.5625, from the same point on scale r_{23}.

(*g*) $1 - .60^2$ Place a straight edge through 60, scale $1/k^2$, and 100, product scale, and read the answer, .64, on scale r_{13}.

(*h*) $\frac{.78 - .60 \times .80}{1 - 80^2}$ Find the product of .60 and .80 by (*a*). On a separate scratch paper subtract this from .78, obtaining .30. Place a straight edge between 30, scale r_{13}, and 80, scale $1/k^2$, and read the answer, .833, on the product scale.

(*i*) $\frac{.78 - .60 \times .80}{\sqrt{1 - 60^2}\sqrt{1 - .80^2}}$ Find $\frac{.78 - .60 \times .80}{1 - .80^2}$ by (*h*).

Find $\frac{.78 - .60 \times .80}{1 - .60^2}$ by (h). Multiply and extract the square root by (a) and (c), yielding the answer .625.

(j) Given: $\beta_{12\cdot4} = .70$; $\beta_{13\cdot4} = 60$; $\beta_{32\cdot4} = .80$; $\beta_{23\cdot4} = .5469$. Required: $\beta_{12\cdot34} = \frac{.70 - .60 \times .80}{1 - .80 \times .5469}$ Find the numerator as in (h) and the denominator in the same manner. Then divide as in (b). This gives $\frac{.2200}{.5625} = .3911$.

If, as is frequently the case, $\beta_{32\cdot4}$ and $\beta_{23\cdot4}$ are nearly equal, $k^2{}_{s\cdot4}$ is closely given by:

$$k^2_{23\cdot4} = 1 - \left(\frac{\beta_{32\cdot4} + \beta_{23\cdot4}}{2}\right)^2.$$

In this case the procedure may be as follows:

$$\frac{.70 - .60 \times .80}{1 - .80 \times .76}$$

Find the numerator, .2200, as before. On scratch paper determine .78, the arithmetic average of .80 and .76. Place a straight edge between .78, scale $1/k^2$, and .22, scale r_{13}, and read the answer, .5618, on the product scale. This answer is in error by .0006, which is of the same order of magnitude as the error attendant upon the use of the large chart.

As a sample problem in three variables the following data are given:

TABLE OF CORRELATIONS, MEANS AND STANDARD DEVIATIONS

	Variables 1	2	3
2	.225		
3	.274	.404	
Means	68.15	43.60	52.20
σ's	10.50	12.24	9.63

Solving

$$\beta_{12\cdot3} = .1366$$
$$\beta_{21\cdot3} = .1236$$
$$\beta_{13\cdot2} = .2200$$
$$k^2_{1\cdot23} = .9093$$
$$r_{1\cdot23} = .3011$$
$$\sigma_{1\cdot23} = 10.01$$

$$\bar{z}_1 = .1366\ z_2 + .2200\ z_3$$

$$\overline{X}_1 = .1172\ X_2 + .2399\ X_3 + 50.52$$

As a sample problem in four variables the following data are given:

	Variables 1	2	3	4
2	.225			
3	.274	.404		
4	.134	.060	.231	
Means	68.15	43.60	52.20	45.40
σ's	10.50	12.24	9.63	14.25

Solving

$$\beta_{12\cdot 34} = .1398$$
$$\beta_{21\cdot 34} = .1270$$
$$\beta_{13\cdot 24} = .1991$$
$$\beta_{14\cdot 23} = .0796$$
$$k^2_{1\cdot 234} = .9033$$
$$r_{1\cdot 234} = .3109$$
$$\sigma_{1\cdot 234} = 9.980$$

$$\bar{z}_1 = .1398\ z_2 + .1991\ z_3 + .0796\ z_4$$

$$\overline{X}_1 = .1199\ X_2 + .2171\ X_3 + .0587\ X_4 + 48.92$$

(*k*) The detailed steps in the solution of the accompanying five-variable problem are given. Such a scheme as shown for recording calculations will facilitate procedure.

TABLE I

VARIABLES

	1	2	3	4	5
2	.72				
3	.62	.61			
4	.58	.61	.66		
5	.63	.82	.55	.59	

TABLE II

FIRST ORDER β'S, ETC.

	.2	.3	.4	.5
12		.5444	.5832	.6209
		.5552	.5518	.3373
				.7906
13	.2879		.4203	.3921
			.3574	.4535
			.8498	.8218

14	.2242	.3026 .2774 .2900		.3195 .3454 .3325
15	.1209	.4143 .4695	.4415 .4337 .4376	
23			.3675 .3303 .3489	.2280 .4853 .8894
24		.3675 .3303 .3489		.1936 .3852 .9254
25		.6946 .7716 .4640	.7058 .7328 .7193	
34	.4585 .4585 .4585			.5146 .4810 .4978
35	.1520 .0793 .9879		.2464 .2845 .2654	
45	.2741 .1430 .9608	.3254 .4022 .8691		

TABLE III

SECOND ORDER β'S, ETC.

$\beta_{14\cdot23} = .1168$	$\beta_{13\cdot25} = .2818$	$\beta_{12\cdot34} = .5058$	$\beta_{12\cdot45} = .5379$
		$\beta_{21\cdot34} = .4948$	$\beta_{21\cdot45} = .3039$
$\beta_{15\cdot23} = .0780$	$\beta_{14\cdot25} = .2154$	Aver. = .5003	
			$k^2_{12\cdot45} = .8365$
$\beta_{45\cdot23} = .2069$	$\beta_{34\cdot25} = .4546$		
$\beta_{54\cdot23} = .1350$	$\beta_{43\cdot25} = .4421$	$\beta_{15\cdot34} = .3634$	$\beta_{13\cdot45} = .3169$
	Aver. = .4483	$\beta_{51\cdot34} = .3907$	$\beta_{31\cdot45} = .3099$
$k^2_{45\cdot23} = .9721$		Aver. = .3771	Aver. = .3134
		$\beta_{25\cdot34} = .6616$	$\beta_{23\cdot45} = .1792$
		$\beta_{52\cdot34} = .7270$	$\beta_{32\cdot45} = .3102$
		$k^2_{25\cdot34} = .5190$	$k^2_{23\cdot45} = .9444$

TABLE IV

THIRD ORDER β'S, ETC.

$\beta_{12\cdot345} = .4655$ $\beta_{13\cdot245} = .2334$ $\beta_{14\cdot235} = .1093$ $\beta_{15\cdot234} = .0554$
$\beta_{21\cdot345} = .2754$

$k^2_{12\cdot345} = .8718$

The constants derived in Tables III and IV give the regression equation:

$$\bar{z}_1 = .4655\ z_2 + .2334\ z_3 + .1093\ z_4 + .0554\ z_5$$

Also

$$k^2_{1\cdot2345} = .4218;\ r_{1\cdot2345} = .7604;$$

$$k_{1\cdot2345} = .6495;\ \sigma_{1\cdot2345} = .6495\ \sigma_1 .$$

In Table II first order β's are given, primary subscripts being shown in the stub, and secondary subscripts in the captions of the columns. The various entries in a compartment of the table have the following meanings: The first entry is the regression coefficient indicated, the second its conjugate, the third the arithmetic average of the two (calculated in case the two coefficients are nearly equal), and the fourth the k^2 derived from the two conjugate β's (calculated in case the two coefficients are quite unequal). Whenever there is a third entry there is no fourth, and vice versa, as but one of these two items is needed.

Table III is derived from Table II, each entry in Table III being calculated in two ways.

Table IV is derived from Table III, each entry being calculated in two ways.