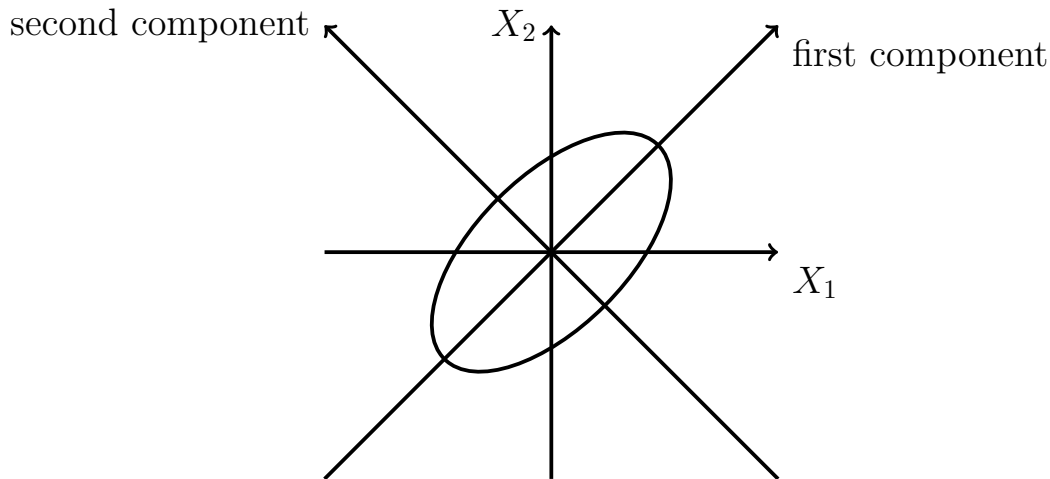


### Principal Component Analysis:

For a geometric interpretation of principal components, suppose we have two variables,  $X_1$  and  $X_2$ , that are centered at their respective means (i.e., the means of the scores on  $X_1$  and  $X_2$  are zero). In the diagram below, the ellipse represents the scatter diagram of the sample points. The first principal component is a line through the widest part; the second component is the line at right angles to the first principal component. In other words, the first principal component goes through the fattest part of the “football,” and the second principal component through the next fattest part of the “football” and orthogonal to the first; and so on. Or, we take our original frame of reference and do a rigid transformation around the origin to get a new set of axes; the origin is given by the sample means (of zero) on the two  $X_1$  and  $X_2$  variables. (To make these same geometric points, we could have used a constant density contour for a bivariate normal pair of random variables,  $X_1$  and  $X_2$ , with zero mean vector.)



As an example of how to find the placement of the components in the picture given above, suppose we have the two variables,  $X_1$  and  $X_2$ , with variance-covariance matrix

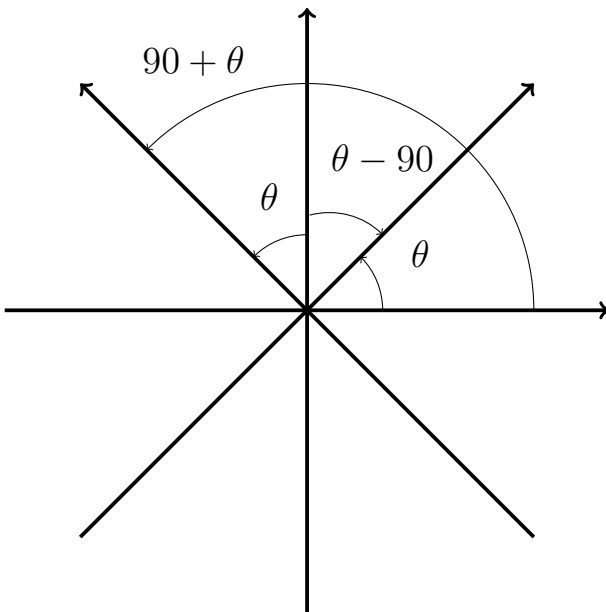
$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} .$$

Let  $a_{11}$  and  $a_{21}$  denote the weights from the first eigenvector of  $\Sigma$ ;  $a_{12}$  and  $a_{22}$  are the weights from the second eigenvector. If these are placed in a  $2 \times 2$

orthogonal (or rotation) matrix  $\mathbf{T}$ , with the first column containing the first eigenvector weights and the second column the second eigenvector weights, we can obtain the direction cosines of the new axes system from the following:

$$\mathbf{T} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \cos(90 + \theta) \\ \cos(\theta - 90) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} .$$

These are the cosines of the angles with the positive (horizontal and vertical) axes. If we wish to change the orientation of a transformed axis (i.e., to make the arrow go in the other direction), we merely use a multiplication of the relevant eigenvector values by  $-1$  (i.e., we choose the other normalized eigenvector for that same eigenvalue, which still has unit length).



If we denote the data matrix in this simple two variable problem as  $\mathbf{X}_{n \times 2}$ , where  $n$  is the number of subjects and the two columns represent the values on variables  $X_1$  and  $X_2$  (i.e., the coordinates of each subject on the original axes), the  $n \times 2$  matrix of coordinates of the subjects on the transformed axes, say  $\mathbf{X}_{trans}$  can be given as  $\mathbf{X}\mathbf{T}$ .

Hotelling's Power method:

From Morrison (1967):

Let  $\mathbf{A}$  be the  $p \times p$  matrix of real elements. It is not necessary that  $\mathbf{A}$  be symmetric. Order the characteristic roots  $\lambda_i$  of  $\mathbf{A}$  by their absolute values:

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_p| ,$$

and denote their respective characteristic vectors as  $\mathbf{a}_1, \dots, \mathbf{a}_p$ . Initially we shall require that only  $|\lambda_1| > |\lambda_2|$ . Let  $\mathbf{x}_0$  be any vector of  $p$  real components, and form the sequence:  $\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0$ ;  $\dots$   $\mathbf{x}_n = \mathbf{A}\mathbf{x}_{n-1} = \mathbf{A}^n\mathbf{x}_0$  of vectors. Then if the successive  $\mathbf{x}_i$  are scaled in some fashion, the sequence of standardized vectors will converge to the characteristic vector  $\mathbf{a}_1$ . Probably the most convenient scaling is performed by dividing the elements by their maximum, with normalization to unit length merely reserved for the last, or exact, vector. Since  $\mathbf{A}\mathbf{a}_1 = \lambda_1\mathbf{a}_1$ , the characteristic root itself can be found by dividing any element of  $\mathbf{A}\mathbf{a}_1$  by the corresponding element of  $\mathbf{a}_1$ .

The same iterative procedure can be used to compute any distinct characteristic root of  $\mathbf{A}$ . To extract the second largest root and its vector we normalize the first characteristic vector  $\mathbf{a}_1$  to unit length, form the  $p \times p$  matrix  $\lambda_1\mathbf{a}_1\mathbf{a}'_1$  and subtract it from  $\mathbf{A}$  to give the residual matrix  $\mathbf{A}_1 = \mathbf{A} - \lambda_1\mathbf{a}_1\mathbf{a}'_1$ .

Kelley's method (from *Essential Traits of Mental Life* (1935)):

If it is desired to create two new variables,  $x'$  and  $y'$ , which are completely defined by the given variables,  $x$  and  $y$ , all that is necessary is to write

$$x' = a_1x + b_1y$$

$$y' = a_2x + b_2y$$

and assign any values to  $a_1$ ,  $a_2$ ,  $b_1$ , and  $b_2$ . Solving these equations for  $x$  and  $y$  we have

$$x = \frac{b_2x' - b_1y'}{a_1b_2 - a_2b_1}$$

$$y = \frac{a_1y' - a_2x'}{a_1b_2 - a_2b_1}$$

Of the infinite number of new sets of equivalent variables,  $x'$  and  $y'$ , which can be derived by substituting different values for  $a_1$ ,  $a_2$ ,  $b_1$ , and  $b_2$ , that one is considered to have special merit which is a rotation of the  $x$  and  $y$  axes to

the position of the major and minor axes of the ellipse. These particular new variables, which we designate  $x_1$  and  $y_1$ , are given by the equations

$$x_1 = x \cos \theta + y \sin \theta$$

$$y_1 = -x \sin \theta + y \cos \theta$$

where  $\theta$  is the angle of rotation and is given by

$$\tan 2\theta = \frac{2p}{v_1 - v_2}$$

[ $p = \sigma_{12}$ ;  $v_1 = \sigma_1^2$ ;  $v_2 = \sigma_2^2$ ] The peculiar merit of the new variables,  $v_1$  and  $y_1$ , lies in the facts which can be immediately surmised by thinking of the elementary geometry involved.

- (a)  $x_1$  and  $y_1$  are uncorrelated.
- (b)  $x_1$  and  $y_1$  axes are at right angles to each other.
- (c) The variance of  $x_1$  distance from the minor axis in the direction of the major axis, is a maximum, for no other rotation of axes yields a variable with as large a variance.
- (d) The variance of  $y_1$ , distance measured in the direction of the minor axis, is a minimum.

The advantage of (a), lack of correlation, need scarcely be dwelt upon, as it is the essential purpose of factorization to obtain independent measures.

The advantage of (b), orthogonality, is not quite so obvious. Though a point in two-dimensional space may be completely defined by distance from two oblique axes, nevertheless the simplicity of thought (and to create such simplicity is a basic purpose of factorization) when a point is defined in terms of perpendicular distance from two perpendicular axes, should be sufficient to commend the use of such axis.

The advantage of (c) making the variance of one of the new variables a maximum is particularly apparent when the major axis is much greater than the minor. In this case, much more about the total situation or the total field wherein variation can take place is known if variability in any other direction is known. The principle of parsimony of thought recommends a knowledge of

the  $x_1$  variable if but a single item of knowledge is available. The operation of this principle will be much more apparent when thinking of many variables, for here the variances of some of the smaller ones may be such that entire lack of knowledge of them will not be serious.

It is obvious from the geometry of the situation that there is but a single solution yielding variables with the properties mentioned. These constitute the components in the two-variable problem.

### **Kelley True Score Prediction (James-Stein Estimation):**

In prediction, two aspects of variable unreliability have consequences for ethical reasoning. One is in estimating a person’s true score on a variable; the second is in how regression might be handled when there is measurement error in the independent and/or dependent variables. In both of these instances, there is an implicit underlying model for how any observed score,  $X$ , might be constructed additively from a true score,  $T_X$ , and an error score,  $E_X$ , where  $E_X$  is typically assumed uncorrelated with  $T_X$ :  $X = T_X + E_X$ . When we consider the distribution of an observed variable over, say, a population of individuals, there are two sources of variability present in the true and the error scores. If we are interested primarily in structural models among true scores, then some correction must be made because the common regression models implicitly assume that variables are measured without error.

The estimation,  $\hat{T}_X$ , of a true score from an observed score,  $X$ , was derived using the regression model by Kelley in the 1920s (Kelley, 1947), with a reliance on the algebraic equivalence that the squared correlation between observed and true score is the reliability. If we let  $\hat{\rho}$  be the estimated reliability, Kelley’s equation can be written as

$$\hat{T}_X = \hat{\rho}X + (1 - \hat{\rho})\bar{X} ,$$

where  $\bar{X}$  is the mean of the group to which the individual belongs. In other words, depending on the size of  $\hat{\rho}$ , a person’s estimate is partly due to where the person is in relation to the group—upward if below the mean, downward if above. The application of this statistical tautology in the examination of group differences provides such a surprising result to the statistically naive that this equation has been labeled “Kelley’s Paradox” (Wainer, 2005, pp. 67–70).

In addition to obtaining a true score estimate from an obtained score, Kelly’s regression model also provides a standard error of estimation (which in this case is now referred to as the standard error of measurement). An approximate 95% confidence interval on an examinee’s true score is given by

$$\hat{T}_X \pm 2\hat{\sigma}_X((\sqrt{1 - \hat{\rho}})\sqrt{\hat{\rho}}) ,$$

where  $\hat{\sigma}_X$  is the (estimated) standard deviation of the observed scores. By itself, the term  $\hat{\sigma}_X((\sqrt{1 - \hat{\rho}})\sqrt{\hat{\rho}})$  is the standard error of measurement, and

is generated from the usual regression formula for the standard error of estimation but applied to Kelly's model predicting true scores. The standard error of measurement most commonly used in the literature is not Kelly's but rather  $\hat{\sigma}_X \sqrt{1 - \hat{\rho}}$ , and a 95% confidence interval taken as the observed score plus or minus twice this standard error. An argument can be made that this latter procedure leads to "reasonable limits" (after Gulliksen, 1950) whenever  $\hat{\rho}$  is reasonably high, and the obtained score is not extremely deviant from the reference group mean. Why we should assume these latter preconditions and not use the more appropriate procedure to begin with, reminds us of a Bertrand Russell quotation (1919, p. 71): "The method of postulating what we want has many advantages; they are the same as the advantages of theft over honest toil."

There are several remarkable connections between Kelley's work in the first third of the twentieth century and the modern theory of statistical estimation developed in the last half of the century. In considering the model for an observed score,  $X$ , to be a sum of a true score,  $T$ , and an error score,  $E$ , plot the observed test scores on the  $x$ -axis and their true scores on the  $y$ -axis. As noted by Galton in the 1880s (Galton, 1886), any such scatterplot suggests two regression lines. One is of true score regressed on observed score (generating Kelley's true score estimation equation given in the text); the second is the regression of observed score being regressed on true score (generating the use of an observed score to directly estimate the observed score). Kelley clearly knew the importance for measurement theory of this distinction between two possible regression lines in a true-score versus observed-score scatterplot. The quotation given below is from his 1927 text, *Interpretation of Educational Measurements*. The reference to the "last section" is where the true score was estimated directly by the observed score; the "present section" refers to his true score regression estimator:

This tendency of the estimated true score to lie closer to the mean than the obtained score is the principle of regression. It was first discovered by Francis Galton and is a universal phenomenon in correlated data. We may now characterize the procedure of the last and present sections by saying that in the last section regression was not allowed for and in the present it is. If the reliability is very high, then there is little difference between [the two methods], so that this second technique, which is slightly the more laborious, is not demanded, but if the reliability is low, there is much difference in individual outcome, and

the refined procedure is always to be used in making individual diagnoses. (p. 177)

Kelley's preference for the refined procedure when reliability is low (that is, for the regression estimate of true score) is due to the standard error of measurement being smaller (unless reliability is perfect); this is observable directly from the formulas given earlier. There is a trade-off in moving to the regression estimator of the true score in that a smaller error in estimation is paid for by using an estimator that is now biased. Such trade-offs are common in modern statistics in the use of "shrinkage" estimators (for example, ridge regression, empirical Bayes methods, James–Stein estimators). Other psychometricians, however, apparently just don't buy the trade-off; for example, see Gulliksen (*Theory of Mental Tests*; 1950); Gulliksen wrote that "no practical advantage is gained from using the regression equation to estimate true scores" (p. 45). We disagree—who really cares about bias when a generally more accurate prediction strategy can be defined?

What may be most remarkable about Kelley's regression estimate of true score is that it predates the work in the 1950s on "Stein's Paradox" that shook the foundations of mathematical statistics. A readable general introduction to this whole statistical kerfuffle is the 1977 *Scientific American* article by Bradley Efron and Carl Morris, "Stein's Paradox in Statistics" (236(5), 119-127). When reading this popular source, keep in mind that the class referred to as James–Stein estimators (where bias is traded off for lower estimation error) includes Kelley's regression estimate of the true score. We give an excerpt below from Stephen Stigler's 1988 Neyman Memorial Lecture, "A Galtonian Perspective on Shrinkage Estimators" (*Statistical Science*, 1990, 5, 147-155), that makes this historical connection explicit:

The use of least squares estimators for the adjustment of data of course goes back well into the previous century, as does Galton's more subtle idea that there are two regression lines. . . . Earlier in this century, regression was employed in educational psychology in a setting quite like that considered here. Truman Kelley developed models for ability which hypothesized that individuals had true scores . . . measured by fallible testing instruments to give observed scores . . . ; the observed scores could be improved as estimates of the true scores by allowing for the regression effect and shrinking toward the average, by a procedure quite similar to the Efron–Morris estimator. (p. 152)

Before we leave the topic of true score estimation by regression, we might



also note what it does not imply. When considering an action for an individual where the goal is to help make, for example, the right level of placement in a course or the best medical treatment and diagnosis, then using group membership information to obtain more accurate estimates is the appropriate course to follow. But if we are facing a contest, such as awarding scholarships, or offering admission or a job, then it is inappropriate (and ethically questionable) to search for identifiable subgroups that a particular person might belong to and then adjust that person's score accordingly. Shrinkage estimators are "group blind." Their use is justified for whatever population is being observed; it is generally best for accuracy of estimation to discount extremes and "pull them in" toward the (estimated) mean of the population.

### From Talents and Tasks; Canonical Correlation Calculations:

Problem 4: To determine that combination of the  $u$  measures and that combination of the  $w$  measures that yields the greatest consociated covariance, thence the second combination of each to yield the greatest consociated covariance in the residual variances, thence with third, fourth, and further combinations. The solution of this problem defines the activity which will yield the greatest happiness to the largest numbers and produce the greatest amount of that which society most needs. The “greatest happiness to the largest numbers” is a loose way of saying that both the degree of happiness and the number affected are involved in the maximizing process. Should one such activity be found, perhaps farming, it is obvious that other activities, enjoyable to lesser numbers and serving lesser social needs, are necessary; so second, third, fourth, etc., types as defined by the subsequent consociated covariances must be found. In spite of the seeming verbal contradiction there is meaning in the statement “Happy is the man whose vocation is his avocation and sound is that society the fulfillment of whose needs is the pleasure of its citizens.”

#### Computational Steps:

The solution of Problem 4 is readily accomplished by successive rotations of axes so as to maximize covariance. For simplicity of explanation let us arrange the given  $w$  variables in an order which we will call  $x_1, \dots, x_w$  such that the sum of the squares of the interaction covariances (those from the upper right quadrant) involving  $x_1$  is greater than that involving  $x_2$ , etc. Similarly, let us arrange the  $u$  variables in an order which we will call  $x_{w+1}, \dots, x_{w+u}$  which is greater than that involving  $x_{w+3}$ , etc.

We now need the general formulas for rotating any two variables  $x_1$  and  $x_2$  through an angle  $\theta$  to obtain new variables  $y_1$  and  $y_2$ . (the bracket notation used below is for the covariance of the included variables) These are

$$y_1 = x_1 \cos \theta + x_2 \sin \theta$$

$$y_2 = -x_1 \sin \theta + x_2 \cos \theta$$

$$[y_1 y_1] = [x_1 x_1](\cos \theta)^2 + [x_2 x_2](\sin \theta)^2 + 2[x_1 x_2](\sin \theta)(\cos \theta)$$

$$[y_2 y_2] = [x_1 x_1](\sin \theta)^2 + [x_2 x_2](\cos \theta)^2 - 2[x_1 x_2](\sin \theta)(\cos \theta)$$

$$[y_1y_2] = [x_1x_2]((\cos \theta)^2 - (\sin \theta)^2) - ([x_1x_1] - [x_2x_2])(\cos \theta)(\sin \theta)$$

$$[y_1x_3] = [x_1x_3] \cos \theta + [x_2x_3] \sin \theta$$

$$[y_2x_3] = -[x_1x_3] \sin \theta + [x_2x_3] \cos \theta$$

If the angle  $\theta$  has been so chosen as to make  $[y_1x_3]$  a maximum, we have

$$\frac{d[y_1x_3]}{d\theta} = -[x_1x_3] \sin \theta + [x_2x_3] \cos \theta = 0$$

or

$$\tan \theta = \frac{[x_2x_3]}{[x_1x_3]},$$

giving  $\theta$  to maximize  $[y_1x_3]$

We note that rotating through this angle makes  $[y_2x_3]$  a minimum, for

$$[y_2x_3] = -[x_1x_3] \sin \theta + [x_2x_3] \cos \theta = 0$$

. Also upon squaring the equation

$$[y_1x_3] = [x_1x_3] \cos \theta + [x_2x_3] \sin \theta$$

we find that for rotation through the angle given by  $\theta$

$$[y_1x_3]^2 = [x_1x_3]^2 + [x_2x_3]^2$$

giving the squared covariance as the sum of squared covariances. This rotation has transferred the squared covariance  $[x_2x_3]^2$  to the (1,3) cell.

### **Kelley's Will:**

The fitter, the richer

Prof's will sets up fitness tests for sons

Santa Barbara, Calif. (AP) – A retired Harvard professor has willed his two sons an extra inheritance—scaled to how they and their future wives rate in a series of mental-physical-character tests.

The unique bequest is from Dr. Truman Lee Kelley, authority on psychological testing and measurement, and one of the authors of the widely used Stanford achievement test battery.

He died May 2 at 76, and terms of his will and its so-called “eugenics trust” were disclosed when it was filed Thursday with the county clerk.

His apparent goal: To obtain for his sons superior marriages through eugenic selection. Eugenics is a process of race improvement by mating superior types suited to each other.

The will sets up a complicated series of tests for his sons, Kenneth, 22, Air Force lieutenant stationed at Wurtsmith AFB, Mich., and Kalon, 24, reportedly working for a Ph.D. degree at the Massachusetts Institute of Technology.

Neither could be reached immediately for comment

Both are single. They and the women they marry are to receive cash awards in proportion to their fitness as determined by the tests. The birth of children to a high-scoring couple calls for additional awards.

Dr. Kelley named a group of trustees headed by Eric F. Gardner of Syracuse University, to devise the test and administer the “eugenics trusts” for his sons, and left 50 shares of Illinois Terminal stock plus other funds to pay the awards.

The trusts were established, the 1954-dated will says, “to promote the eugenic marriage of my sons through counseling, marriage and birth awards.”

To be eligible for a marriage award the son and his prospective bride must give the trustees any information they need to determine prior to marriage the couple's “E” (for eugenics) score.

The “E” score would be set by the couple's deviation about or below the American white population average in three categories: health, intellect, and character.

For each point about average, the son and his bride would receive \$400

each, from which would be deducted \$400 for each point below average. On the birth of a child, a similar award of \$600 per point would be paid.

Since the number of possible points still is to be determined by the trustees, the highest amount available was not stated. The total value of property owned by Dr. Kelley was estimated at \$175,000.

In addition to the fund set up to finance the eugenics trusts, Dr. Kelley left half his estate to his widow, Grace, and one-fourth each to the sons.

May 12, 1961

Father's will prods sons to improve human race

Trust fund to reward future weddings if science approves

Santa Barbara, Calif., May 12 (AP) – A Harvard psychologist who died after a lifetime devoted to studying the human race has left a will devoted to improving it.

Method: Cash bonuses for his two sons. Condition: They and the wives they select must score well on a series of mental-physical-character tests.

The will, admitted to probate yesterday, got this reaction from the widow of Truman Lee Kelley:

“We pay so much attention to our prize cattle, but we haven't paid too much attention to the human race.”

Said Mrs. Kelley: “This is a step in the right direction.”

Sons Say Yes and No

But Dr. Kelley's two sons had varying reactions.

“I see some disadvantages in it,” said his son Kenneth, 21, an air force second lieutenant. “For one thing, I don't want to select a wife solely on the standards he imposed in his will. I don't want to place money before marriage itself.”

The other son, Kalon, 24, a staff member at Massachusetts Institute of Technology said he agrees with terms of the will and “very definitely intends to comply” he added.

“What my father tried to do when he instituted and authored the will was to instill in my brother and myself a consideration of eugenic aspects he and I consider important.”

(Eugenics is the study of selective breeding.)

### No Marriage Plans

Neither son plans to marry in the near future, they said. Both said they discussed the unusual will with their father, who died last May at the age of 76.

Dr. Kelley achieved fame in 1923 as one of the authors of the widely used Stanford achievement test battery. He served the Government as a psychological testing authority in World War II. After his retirement from Harvard about 12 years ago, he and his family moved to this southern California seaside city.

The will estimates his total worth at about \$175,000.

His will, written in 1954, provides for two trust funds: one for his sons and one for other young persons whose marriages, he believed, would result in superior children.

### Trust to Help PhDs

Mrs. Kelley indicated that the second trust fund was primarily for college graduates seeking advanced degrees she explained.

“He always was bothered by the fact that candidates for the PhD degree had to choose between getting the degree or marrying and having children.”

“Usually they married and delayed having children, so he thought that in some small way he might enable these students to continue their studies and have youngsters.”

Dr. Kelley appointed a group of trustees, headed by Eric F. Gardner of Syracuse University, to administer the “eugenics trusts.”

The amount of each trust fund, based on stock holdings, was not disclosed.

### Test For Candidates

To qualify for the money, the sons and their prospective brides – before marrying – must give the trustees information required to determine the couple’s “E” (for eugenics) score.

The “E” score would be based on the pair’s deviation above or below the American white population averages for health, intellect and character.

Each point above average would bring the son and his bride \$400; each point below average would result in a \$400 deduction.

The couples would get \$600 for each child.

The maximum amount available will be determined by the trustees.

**Part of an interview with Gene Golub by Thomas Haigh for SIAM (2005):**

HAIGH: Actually, I'm seeing you mentioned here. So this is Progress Report #9 covering January 1, 1954 to June 30, 1954. The report says that you were a half-time research assistant, and it says, "Mr. Golub has been studying the problems associated with factor analysis and is also working on other problems associated particularly with matrix operations. He has programmed Rao's Maximum Likelihood Factor Analysis Method, and has obtained new results, which he will publish. This summer, he presented a paper on tests of significance in factor analysis at the International Psychological Congress in Montreal, and in September will present another at the meeting of the American Psychological Association in New York."

GOLUB: That was very interesting. One of the groups that was very active at Illinois was the psychologists, or psychometricians, and they had a group of first-rate people there. I got to know a man by the name of Charles Wrigley quite well. He was a psychologist. I guess he was on a research appointment. He was from New Zealand originally. Wrigley got his Ph.D. in London under a man by the name of Cyril Burt; he was the originator of the Eleven Plus exam. He has been subsequently discredited because it appears that some of his research data on twins reared apart was made up.

HAIGH: The identical twins that no one could find. Was that him?

GOLUB: In this case, it was a secretary that no one could find. Anyway, Wrigley worked for him, and he was a very powerful man and intellectual man, although rather sloppy in his presentations. Wrigley had a job in Montreal when he came to North America after he got his Ph.D. Then he was being looked over at Illinois. When he saw the ILLIAC, he didn't want to do any further numerical computations. I guess he had used an old hand calculator, and he saw this machine that could do all these computations. The guys at Illinois, the psychologists who were there, they knew they needed somebody who had some experience in numerical computing, so they hired him. He was an inspiration to me. He was so positive and helpful. He was just a very nice man and invited people to his house—just a terrifically good human being.

So I got in with that factor analysis group, and I learned a lot. It was really important for me subsequently. Nowadays, there's this whole tendency

to look at larger and larger data sets. It's interesting. The people who first started this, I think, were the psychologists. They had large sets of data coming out of educational testing and so forth. So they were the ones that first began this whole interest in analyzing large data sets. Factor analysis is the term used for that, and they were very involved. Illinois was a hot center of this because of the people that they had, plus, they had a big Air Force contract. So this area of research, of data analysis, went on. For instance, electrical engineers had been involved in that. And then finally, I was saying in the last ten years, computer scientists began to realize that they're getting masses of data, and then the question is how to analyze this data and what to do with it. It started me off in an area that has been of great interest to me. Simultaneously, when I was a graduate student, there was a man by the name of C.R. Rao. He's probably one of the world's great statisticians. He developed this technique for canonical factor analysis. He's still alive at Penn State University, and I'm going to actually visit him next summer.

HAIGH: Were there a number of Ph.D. students?

GOLUB: There were a few around. So the idea was that I was going to continue in statistics, and I was going to write a thesis, possibly, under a man called Bill Madow. So the sequence of events is: the first year, I was a graduate student. Madow was on sabbatical, and C.R. Rao was at Illinois. The next year Rao left and Madow came back, but he came back without his wife. They had been in California on a sabbatical. He was not a well-tempered person. He was a very smart guy, but he was very difficult. Then after the year, he left to go back to California. So this guy, Madow, ended up here in Northern California, working for SRI. All right. Then (Abraham) Taub suddenly grabs a hold of me, so I could be his student. So he took me on as a student.

HAIGH: Was Taub head of the lab at that point?

GOLUB: Well, as soon as Nash left. I don't know what year Nash left. Taub became the head. So I was supposed to be Taub's student. He said, "Here, read this," and there was some paper he gave me by von Neumann that hadn't been published yet. So I looked at it. I didn't really see how to go from there, but eventually, that paper played an enormous role in what I did. So I started to work on that. He didn't know that field so well himself. He



just put me into it. So I was no longer doing statistics, although I was getting my degree in statistics, I was doing numerical analysis. In the meantime, I was subject to a lot of abuse by Taub. He would just yell and scream at me. Did [Bill] Gear talk to you about this at all?

HAIGH: Yes.

GOLUB: I think he liked Gear better. Maybe Gear is a more secure and sophisticated person. I think the weaker you are as a person, the more he hammered at you. He was really a nasty piece. At any rate, I finished the thesis.

HAIGH: Had work began on ILLIAC 2 while you were there?

GOLUB: Yes, there was talk about it in the background, but I didn't participate. There was another student amongst the people. His name is Roger Farrell, and we've remained very good friends. He's now a retired professor of statistics at Cornell University. Roger was really a statistician. He got his degree under Burkholder at Illinois. So it was just a good place and things were expanding. With respect to new ideas coming out of Illinois in terms of computation, Gregory had programmed the Jacobi method for computing eigenvalues. He wrote an article about that, which alerted a lot of people to the use of the Jacobi method. It was a method that von Neumann and Goldstein had advocated. Then Gregory wrote this little short-page paper showing what accuracy it obtained, and that pricked up a lot of ears to the use of the Jacobi method.

But there wasn't a lot of innovation in numerical methods. That's what I'm trying to say. There wasn't enough leadership. I once got into an argument with Taub at Bill Gear's wedding party about the fact that there were no real numerical analysts at Illinois. [laughs] Maybe that's what led to his being angry with me, I don't know. Our relationship went up and down, and there were times even after that it was up, but then I think at the end it was down again for various reasons. He died about four or five years ago.

**From an interview with Darrell Bock in JEBS (2006) :**

I had heard from Charles Wrigley at Michigan State University that the new Illiac electronic computer at Champaign-Urbana had programs for both the one- and two-matrix eigenproblems. On his advice, I phoned Kern Dickman, who had helped Charles perform a principal component analysis on the machine, and explained my needs. He invited me to come down to Urbana and bring the matrices to be analyzed with me. By that time, I had become sufficiently proficient in using punched card equipment in the business office of the University—in particular a new electronic calculating punch that could store constants and performed cumulative multiplications as fast as the cards passed through the machine. I used this machine to convert Likert scores of the categories into scale values by a method I had described previously in my 1956 *Psychometrika* paper, “Selection of Judges for Preference Testing.”

I arrived in Urbana and found Kern; he took me directly to the computation Center to see the Illiac. But there was very little to see—only a photoelectric reader of teletype tape and a box with a small slit where punched tape spewed from the machine; a few dimly revealed electronic parts could be seen behind a plate-glass window. Elsewhere in the room were teletype machines for punching numbers and letters onto paper tape, printing out the characters of an existing tape, or copying all or parts of one tape to another. My first job was to key the elements of the two covariance matrices onto tape, which in spite of my best efforts to avoid errors, took most of the afternoon.

When I finished that task, Kern suggested that we should meet for dinner at his favorite watering hole in Urbana. When I arrived there I found him sitting with another person whom he introduced as Gene Golub, adding that Gene had programmed the eigenroutines for the Illiac. At Kern’s suggestion Gene had brought along some papers for me—an introduction to programming the Illiac and the documentation of the eigenroutines. He said that his code was similar to that of Goldstein, who had programmed the eigen-procedures for the Maniac machine built by Metropolis at Los Alamos. It used the Jacobi iterative method, which consists of repeated orthogonal transformations of pairs of variables to reduce the elements in the off-diagonal of a real symmetric matrix to zero, all the while performing the same operation on an identity matrix. Although a given element of the matrix does not necessarily remain

zero, the iterations converge to a diagonal matrix containing the eigenvalues, and the identity matrix becomes the corresponding eigenvectors.

Gene told the story that Goldstein, having heard the Jacobi method described by a colleague, stopped by John von Neumann's office to ask if the method was strictly convergent. Gazing at the ceiling for about five seconds, von Neumann replied "yes, of course." Goldstein was amazed, thinking this was another of von Neuman's fabled feats of mental calculation, but as Golub and Van Loan show in their 1996 reference, *Matrix Computations*, the proof requires only a few lines of matrix expressions, which von Neumann could have easily visualized. I already knew of this method, not as Jacobi's, but as the "method of sine and cosine transformations" described by Truman Kelley in his 1935 book, *Essential Traits of Mental Life*. He presented the method as his own creation, including a proof of convergence requiring several pages of geometric argument. Considering that Jacobi had introduced the method in the middle of the 19th-century, I wondered if Kelley had heard of it from one of his fellow professors at Harvard. But I found in his 1928 book, *Crossroads in the Mind of Man*, that he had already used sine and cosine transformations in connection with Spearman's one-factor model, and I now believe that he rediscovered Jacobi's method independently.

Next morning, Kern showed me how to feed my tape into the reader, and after about 20 minutes I saw the output tape race out of the machine and fall into a large wastebasket that was sitting below. My job was then to pull the free rear end of the tape over to a geared-up hand-cranked reel and quickly pulled the rest of the tape out of the basket. I took the roll of tape to the teletype machine and printed the incredibly rewarding twelve-variable eigenvalues and eigenvectors of my two-matrix eigen-problem.

Later than day, after profuse thanks to Kern, I caught the northbound City of New Orleans back to Chicago. While perusing the eigen-routine documentation during the trip, I saw that Golub had reduced the two-matrix problem to a one-matrix problem by pre- and post-multiplying the first matrix by the inverse of the so-called "grammian square root" of the second matrix. The resulting matrix is symmetric. The grammian square root is a diagonal matrix containing the square roots of the eigenvalues (which must be positive), pre-and post-multiplied by the matrix of eigenvectors and its transpose. If

the diagonal matrix contains the reciprocal square roots of the eigenvalues, the result is the inverse matrix. This method has the disadvantage, however, of requiring the calculation of the eigenvalues and vectors of two symmetric matrices rather than one. But it occurred to me that the computations could be shortened considerably by pre- and post-multiplying by the inverse of the so-called “false square root” of the symmetric positive definite matrix—that is, by the inverse of the triangular Cholesky decomposition of the matrix. For some time I had been solving least-squares regression problems by Cholesky decomposition rather than less accurate conventional Gaussian elimination.

This quicker method of solving the two-matrix eigen-problem is in the MATCAL subroutines programmed by Bruno Repp and me in 1974; the resulting one-matrix eigen-problem is solved by the Householder-Ortega-Wilkinson method as programmed by Richard Wolfe. From the Illiac programming manual I learned a truth that has proved invaluable to me: it said, “Remember that the best programmer is not necessarily the one who makes the fewest mistakes, but one who can find his mistakes most quickly.” I soon learned that as a programmer, I was one of the latter. Lyle wrote up the Illiac results in the paper, “Multivariate Discriminant Analysis Applied to ‘Ways to Live’ ratings from Six Cultural Groups,” which subsequently appeared in the June 1960, issue of *Sociometry*, then published by the American Sociological Association. (The paper is also now online at JSTOR.) To the best of my knowledge it is the first application of multiple discriminant analysis on as many as 12 variables. It paved the way for my later work on multivariate analysis of variance and covariance.