14 Unfolding

The unfolding model is a model for preferential choice. It assumes that different individuals perceive various objects of choice in the same way but differ with respect to what they consider an ideal combination of the objects' attributes. In unfolding, the data are usually preference scores (such as rank-orders of preference) of different individuals for a set of choice objects. These data can be conceived as proximities between the elements of two sets, individuals and choice objects. Technically, unfolding can be seen as a special case of MDS where the within-sets proximities are missing. Individuals are represented as "ideal" points in the MDS space so that the distances from each ideal point to the object points correspond to the preference scores. We indicate how an unfolding solution can be computed by the majorization algorithm. Two variants for incorporating transformations are discussed: the conditional approach, which only considers the relations of the data values within rows (or columns), and the unconditional approach, which considers the relations among all data values as meaningful. It is found that if transformations are allowed on the data, then unfolding solutions are subject to many potential degeneracies. Stress forms that reduce the chances for degenerate solutions are discussed.

14.1 The Ideal-Point Model

To introduce the basic notions of *unfolding* models, we start with an example. Green and Rao (1972) asked 42 individuals to rank-order 15 breakfast

TABLE 14.1. Preference orders for 42 individuals on 15 breakfast items (Green & Rao, 1972). The items are: A=toast pop-up; B=buttered toast; C=English muffin and margarine; D=jelly donut; E=cinnamon toast; F=blueberry muffin and margarine; G=hard rolls and butter; H=toast and marmalade; I=buttered toast and jelly; J=toast and margarine; K=cinnamon bun; L=Danish pastry; M=glazed donut; N=coffee cake; O=corn muffin and butter.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	13	12	7	3	5	4	8	11	10	15	2	1	6	9	14
2	15	11	6	3	10	5	14	8	9	12	7	1	4	2	13
3	15	10	12	14	3	2	9	8	7	11	1	6	4	5	13
4	6	14	11	3	7	8	12	10	9	15	4	1	2	5	13
5	15	9	6	14	13	2	12	8	7	10	11	1	4	3	5
6	9	11	14	4	7	6	15	10	8	12	5	2	3	1	13
7	9	14	5	6	8	4	13	11	12	15	7	2	1	3	10
8	15	10	12	6	9	2	13	8	7	11	3	1	5	4	14
9	15	12	2	4	5	8	10	11	3	13	7	9	6	1	14
10	15	13	10	7	6	4	9	12	11	14	5	2	8	1	3
11	9	2	4	15	8	5	1	10	6	7	11	13	14	12	3
12	11	1	2	15	12	3	4	8	7	14	10	9	13	5	6
13	12	1	14	4	5	6	11	13	2	15	10	3	9	8	7
14	13	11	14	5	4	12	10	8	7	15	3	2	6	1	9
15	12	11	8	1	4	7	14	10	9	13	5	2	6	3	15
16	15	12	4	14	5	3	11	9	7	13	6	8	1	2	10
17	7	10	8	3	13	6	15	12	11	9	5	1	4	2	14
18	7	12	6	4	10	1	15	9	8	13	5	3	14	2	11
19	2	9	8	5	15	12	7	10	6	11	1	3	4	13	14
20	10	11	15	6	9	4	14	2	13	12	8	1	3	7	5
21	12	1	2	10	3	15	5	6	4	13	7	11	8	9	14
22	14	12	10	1	11	5	15	8	7	13	2	6	4	3	9
23	14	6	1	13	2	5	15	8	4	12	7	10	9	3	11
24	10	11	9	15	5	6	12	1	3	13	8	2	14	4	7
25	15	8	7	5	9	10	13	3	11	6	2	1	12	4	14
26	15	13	8	5	10	7	14	12	11	6	4	1	3	2	9
27	11	3	6	14	1	7	9	4	2	5	10	15	13	12	8
28	6	15	3	11	8	2	13	9	10	14	5	7	12	1	4
29	15	7	10	2	12	9	13	8	5	6	11	1	3	4	14
30	15	10	7	2	9	6	14	12	8	11	5	3	1	4	13
31	11	4	9	10	15	8	6	5	1	13	14	2	12	3	7
32	9	3	10	13	14	11	1	2	4	5	15	6	7	8	12
33	15	8	1	11	10	2	4	13	14	9	6	5	12	3	7
34	15	8	3	11	10	2	4	13	14	9	6	5	12	1	7
35	15	6	10	14	12	8	2	4	3	5	11	1	13	7	9
36	12	2	13	11	9	15	3	1	4	5	6	8	10	7	14
37	5	1	6	11	12	10	7	4	3	2	13	9	8	14	15
38	15	11	7	13	4	6	9	14	8	12	1	10	3	2	5
39	6	1	12	5	15	9	2	7	11	3	8	10	4	14	13
40	14	1	5	15	4	6	3	8	9	2	12	11	13	10	7
41	10	3	2	14	9	1	8	12	13	4	11	5	15	6	7
42	13	3	1	14	4	10	5	15	6	2	11	7	12	8	9

items from 1 (= most preferred) to 15 (= least preferred). They obtained the data in Table 14.1, where each row i contains the ranking numbers assigned to breakfast items A, . . ., O by individual i. These numbers express some kind of closeness, the proximity of each item to an optimal breakfast item.

In contrast to other examples discussed so far, the row entries of this matrix differ from the column entries: the former are individuals, the latter breakfast items.[1] It is possible, though, to conceive of Table 14.1 as a *submatrix* of the familiar proximity matrix. This is shown in Figure 14.1, where the shaded rectangles stand for the observed scores. Both rectangles contain the same scores: the rows and columns of one rectangle appear

[1] A matrix with different row and column entries is called a *two-mode* matrix, see Section 3.7.

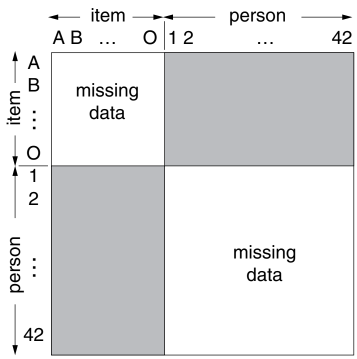

FIGURE 14.1. Schematic view of proximity matrix in Table 14.1 as a submatrix of a complete proximity matrix.

as columns and rows in the other. Each rectangle is called an *off-diagonal corner matrix*. One notes that in this data matrix only *between-sets proximities* are given and no *within-sets proximities*. Hence, one can analyze these proximities by "regular" MDS if the within-sets proximities are treated as missing values.

Ideal Points and Isopreference Contours

Figure 14.2 presents such an unfolding solution for Table 14.1. The resulting configuration consists of 57 points, 42 for the individuals (shown as stars) and 15 for the breakfast items (shown as solid points). Every individual is represented by an *ideal point*. The closer an *object point* lies to an ideal point, the more the object is preferred by the respective individual. For example, Figure 14.2 says that individual 4 prefers K (cinnamon bun) and L (Danish pastry) the most, because the object points of these breakfast items are closest to this individual's ideal point. The circles around point 4 are *isopreference contours*. Each such contour represents a class of choice objects that are preferred equally by individual 4. We note that for individual 4, D and M are slightly less preferred than K and L. Somewhat less preferred is the coffee and cake breakfast (N), whereas A, B, C, E, F, G, H, I, J, and O are more or less equally disliked.

In this way, the preferences for every individual are modeled by relating ideal points to the points representing the choice objects. This defines the *ideal-point model* of unfolding. Note that the model assumes that all individuals share the same psychological space for the choice objects. Individual differences are modeled exclusively by the different ideal points.

The term "unfolding" (Coombs, 1950) was chosen for the following reason. Assume that Figure 14.2 was printed on a thin handkerchief. If this handkerchief is picked up with two fingers at the point representing in-

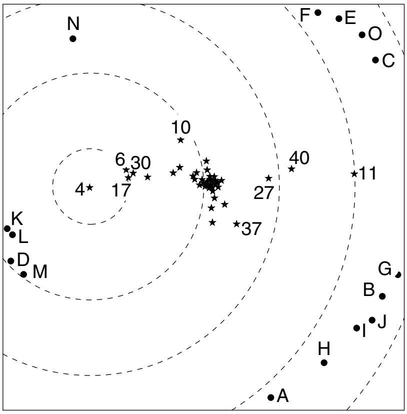

FIGURE 14.2. Unfolding representation of data in Table 14.1. Stars are individuals, solid points are items; the circles show the isopreference contours for individual 4.

dividual i, y_i, and then pulled through the other hand, we have folded it: point y_i is on top, and the farther down the object points, the less preferred the objects they represent. The order of the points in the vertical direction corresponds (if we folded a perfect representation) to how individual i ordered these objects in terms of preference. Picking up the handkerchief in this way at any individual's ideal point yields this individual's empirical rank-order. The MDS process, then, is the inverse of the folding, that is, the unfolding of the given rank-orders into the distances.[2]

Figure 14.2 seems to indicate that none of the breakfast items is particularly attractive to the respondents, because none really comes close to an ideal point (a "star"). Furthermore, we also see that the ideal points scatter quite a bit, indicating considerable interindividual differences in what kind of breakfast item the respondents prefer. Thus, it would be impossible to please everybody with any particular small set of breakfast items. However, before embarking on further interpretations, we should first ask to what extent we can really trust what we see here in the unfolding configuration.

Unfolding: Technical Challenges

An MDS analysis of an off-diagonal proximity matrix poses technical challenges. A lot of data are missing and, moreover, the missing data are not just randomly scattered throughout the data matrix. What does that mean in terms of the model? Consider a case suggested by Green and Carmone (1970). Figure 14.3 shows 35 points, arranged to form an A and an M. Assume that we compute the distances for this configuration, and use them

[2]More precisely, the case just described is conditional unfolding; see below.

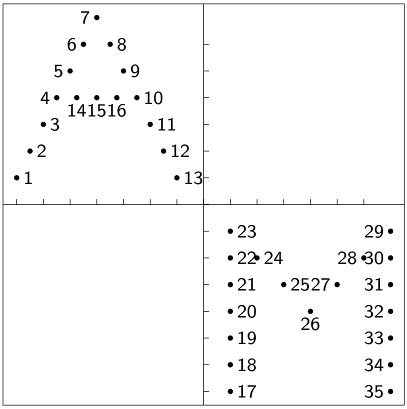

FIGURE 14.3. Synthetic AM configuration (after Green & Carmone, 1970).

as data for ordinal MDS. If a 2D representation is computed, it will, no doubt, recover the underlying AM configuration almost perfectly. But what happens in the unfolding situation when only those data that correspond to distances *between* the points in the A and the M are employed? If M's points are fixed, then, for example, the order of $d(13, 23)$ to $d(13, 29)$ implies that point 13 must be placed to the left of the perpendicular through the midpoint of the line segment connecting 23 to 29. At the same time, the points in A impose constraints on those in M, and, indeed, those are the only ones imposed on M's points, just as M's points are the only points to constrain the points of A. Note that this involves all distances between A and M. Considering that there are many such order relations, it seems plausible to expect a very good recovery of the AM configuration.

In the next sections, we show, however, that blind optimization of Stress (with admissible transformation of the proximities) yields degenerate solutions for unfolding. We discuss why this is so.

14.2 A Majorizing Algorithm for Unfolding

Assume that the proximities are dissimilarities and that no transformations are allowed on the data. Let $\mathbf{W}$ be the partitioned matrix of weights w_{ij},

$$\begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{12}' & \mathbf{W}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{W}_{12} \\ \mathbf{W}_{12}' & \mathbf{0} \end{bmatrix},$$

and let the coordinate matrix $\mathbf{X}$ be partitioned in $\mathbf{X}_1$ for the n_1 individuals and $\mathbf{X}_2$ for the n_2 objects in the unfolding analysis. Because the within-sets proximities are missing, $\mathbf{W}_{11} = \mathbf{0}$ and $\mathbf{W}_{22} = \mathbf{0}$. This weight matrix can be used in any program for MDS that allows missing values to do unfolding.

Heiser (1981) applied this idea for the majorizing algorithm for minimizing Stress (see Chapter 8). The corresponding algorithm is summarized below.

Consider the minimization of raw Stress; that is,

$$\begin{aligned} \sigma_r(\mathbf{X}) &= \sum_{i<j} w_{ij}(\delta_{ij} - d_{ij}(\mathbf{X}))^2 \\ &= \eta_\delta^2 + \text{tr } \mathbf{X}'\mathbf{V}\mathbf{X} - 2\text{tr } \mathbf{X}'\mathbf{B}(\mathbf{X})\mathbf{X}, \end{aligned}$$

where $\mathbf{V}$ is defined as in (8.18) and $\mathbf{B}(\mathbf{X})$ as in (8.24). For the moment, assume that all between-sets weights are one, so that the $n_1 \times n_2$ matrix $\mathbf{W}_{12} = \mathbf{1}\mathbf{1}'$, where the vectors $\mathbf{1}$ are of appropriate lengths. Then, the partitioned matrix $\mathbf{V}$ equals

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{12}' & \mathbf{V}_{22} \end{bmatrix} = \begin{bmatrix} n_2\mathbf{I} & -\mathbf{1}\mathbf{1}' \\ -\mathbf{1}\mathbf{1}' & n_1\mathbf{I} \end{bmatrix}.$$

The majorization algorithm of Section 8.6 proves that Stress is reduced by iteratively taking the Guttman transform (8.28), $\mathbf{X}^u = \mathbf{V}^+\mathbf{B}(\mathbf{Y})\mathbf{Y}$, where $\mathbf{Y}$ is the previous estimate of $\mathbf{X}$. Heiser (1981) showed that for unfolding with equal weights $\mathbf{W}_{12} = \mathbf{1}\mathbf{1}'$ we can use instead of the Moore–Penrose inverse $\mathbf{V}^+$ a generalized inverse

$$\mathbf{V}^- = \begin{bmatrix} n_2^{-1}(\mathbf{I} - n^{-1}\mathbf{1}\mathbf{1}') & \mathbf{0} \\ \mathbf{0} & n_1^{-1}(\mathbf{I} - n^{-1}\mathbf{1}\mathbf{1}') \end{bmatrix},$$

where $n = n_1 + n_2$. $\mathbf{B}(\mathbf{Y})$ can be partitioned in the same way as $\mathbf{V}$; that is,

$$\mathbf{B}(\mathbf{Y}) = \begin{bmatrix} \mathbf{B}_{11}(\mathbf{Y}) & \mathbf{B}_{12}(\mathbf{Y}) \\ \mathbf{B}_{12}(\mathbf{Y})' & \mathbf{B}_{22}(\mathbf{Y}) \end{bmatrix};$$

see (8.24).

The update becomes

$$\mathbf{X}_1^u = [\mathbf{V}^-]_{11}[\mathbf{B}_{11}(\mathbf{Y})\mathbf{Y}_1 + \mathbf{B}_{12}(\mathbf{Y})\mathbf{Y}_2], \quad (14.1)$$

$$\mathbf{X}_2^u = [\mathbf{V}^-]_{22}[\mathbf{B}_{12}(\mathbf{Y})'\mathbf{Y}_1 + \mathbf{B}_{22}(\mathbf{Y})\mathbf{Y}_2]. \quad (14.2)$$

As with every majorizing algorithm, the Stress is reduced in every iteration until convergence is reached.

If the between-sets weights have different values, then the update formulas (14.1) and (14.2) do not work anymore. Instead, the update formula (8.28) for MDS with weights should be applied. The SMACOF algorithm needs the computation of the Moore-Penrose inverse $\mathbf{V}^+$ of the $(n_1 + n_2) \times (n_1 + n_2)$ matrix $\mathbf{V}$ which can be computed outside the iteration loop and stored in memory. For reasonable-sized unfolding problems, the memory and computational effort do not pose a problem for current computers.

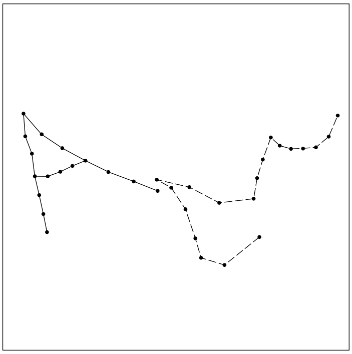

FIGURE 14.4. Ordinal unconditional unfolding representation based on distances between points in A and M in Figure 14.3.

14.3 Unconditional Versus Conditional Unfolding

We now take the 19×16 corner matrix of the between-sets distances of the AM example and check whether ordinal MDS (called "unfolding" under these circumstances) can recover the AM configuration in Figure 14.3. We emphasize ordinal unfolding, but any of the transformations discussed in Chapter 9 for complete MDS can be used.

Unconditional Unfolding

In ordinary MDS, any nonmissing proximity can be compared *unconditionally* to any other nonmissing proximity. For unfolding, this situation is called *unconditional* unfolding.

The unconditional unfolding solution for the AM data is shown in Figure 14.4.[3] Contrary to expectation, this is not a particularly good reconstruction of the original AM configuration. The M is quite deformed, and the A is sheared to the left. Yet, the Stress is only .01, so it seems that the ordinal relations of the between-sets proximities are too weak to guarantee perfect recovery of the underlying configuration.

An MDS analysis for a complete set of proximities on A and M is constrained by many more order relations than doing MDS on an off-diagonal submatrix. In the off-diagonal submatrix, we have $n_A \cdot n_M$ entities, where $n_A = 16$, the number of points in A, and $n_M = 19$ for M. Because we can compare any two entities, we have $\binom{n_A \cdot n_M}{2} = 46,056$ order relations. In

[3]We used the program Minissa-I (Lingoes, 1989), but any other MDS program that allows for missing data could be used as well. Unconditional unfolding can be accomplished by embedding the corner matrix into a complete matrix as shown in Figure 14.1. Programs that allow the user to input off-diagonal matrices directly are only more convenient, but they yield the same solutions as "regular" MDS with missing data.

the complete case (no missing data), we have $\binom{n_A+n_M}{2} = 595$ different entities, and, thus, $\binom{595}{2} = 176,715$ order relations. Yet, the sheer reduction of proximities and thus of relevant order relations between the proximities is, by itself, not of crucial importance: Figure 6.1, for example, shows that almost perfect recovery of the underlying configuration is still possible even when 80% of the proximities are eliminated. This recovery, however, depends critically on a systematic interlocking of the nonmissing proximities. In the unfolding case, such interlocking is not given: rather, there are *no* proximities at all for determining the distances within the two subsets of points.

Conditional Unfolding

The data information is now reduced even further by treating the 19×16 proximity matrix derived from the AM configuration *row-conditionally.*[4] A proximity is only compared to other proximities within its own row, not to proximities in other rows. With $n_A = 16$ and $n_M = 19$, row-conditionality reduces the number of order constraints in the MDS representation from 46,056 in the unconditional case to only $n_A \cdot [n_M(n_M - 1)/2] = 2{,}736$ in the conditional case. An early reference of the use of row-conditional unfolding is Gleason (1967).

Why do we consider *conditional unfolding* at all? After all, the unconditional approach already has serious problems. But consider Table 14.1. Each of its rows is generated by a different individual. For such data, it must be asked whether they can be meaningfully compared over individuals. By comparing the ranks unconditionally, we would assume that if individual i ranks beakfast item x higher than individual j ranks item y, then x comes closer to i's ideal item than y is to j's ideal. This is a strong assumption, because individuals i and j may carry out their ranking task completely differently. For example, i may be essentially indifferent to all items, whereas j likes all items very much so that it becomes difficult to decide which one he or she likes best. In unconditional ordinal unfolding, *all* 1s must be mapped into distances smaller than those representing 2s, and so on, but the row-conditional case requires only that a 1 in a given row is mapped into a distance smaller than the distance representing a 2 of the *same* row, and so on, *for all rows separately.*

For the breakfast item preferences in Table 14.1, the configuration in Figure 14.2 was obtained by ordinal row-conditional unfolding (with the program SSAR-2). The alienation coefficient of this solution is $K = .047$, so the order of the proximities in each row of data seems to match the order

[4]This restriction is called *split-by-rows* by Kruskal and Carmone (1969), which suggests that the data matrix is treated as if we had cut it into horizontal strips: the elements can be compared within a strip, but not between strips.

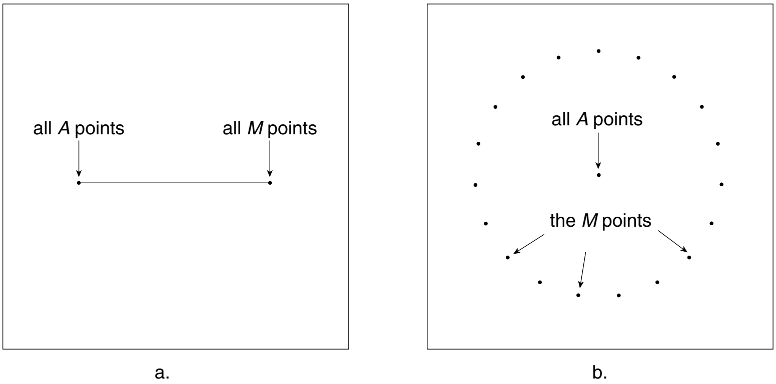

FIGURE 14.5. Trivial unconditional ordinal unfolding solutions for the AM data when using Stress.

of the corresponding distances very well. For individual 11, for example, we find that $d(11, G)$ is indeed the smallest distance, whereas $d(11, D)$ is the greatest distance, corresponding to the ranks 1 for item G and 15 for D. Moreover, the distances from point 11 to those points representing items of intermediate preference are also approximately in agreement with the data. The configuration suggests further that the individuals seem to divide the items into four groups. Yet, we notice that the object points are essentially all located on a circle. Such peculiar regularities often indicate degeneracies in the MDS solution. We turn to this question in the next section.

14.4 Trivial Unfolding Solutions and σ_2

The minimization of Stress for conditional or unconditional unfolding leads easily to trivial or even degenerate solutions, apart from the degeneracies that can occur in ordinary MDS.

The Equal Distance Solution

In unconditional ordinal unfolding there exist two trivial or degenerate solutions if Stress is used as a minimization criterion. That is, Stress can be reduced arbitrarily close to 0, irrespective of the order relations in the data. Two such degenerate solutions are presented in Figure 14.5.

For our AM problem, one trivial solution consists of only two point clusters: all points of the A are condensed into one point and all points of the M into another; the A and the M clusters are clearly separated from each other. The other trivial solution consists of all M points on a circle (not necessarily equally spaced) and the A points in the center, or vice versa. In higher dimensions, the M points could appear on the surface of a (hyper)sphere. These two solutions share the fact that all distances from the ideal points to the object points are the same.

Why these configurations represent solutions to the scaling problem follows from the Stress-1 function, that is, from

$$\sigma_1(\mathbf{X}) = \left(\frac{\sum_{i<j} w_{ij}(d_{ij}(\mathbf{X}) - \widehat{d}_{ij})^2}{\sum_{i<j} w_{ij} d_{ij}^2(\mathbf{X})} \right)^{1/2}, \text{ for all defined } p_{ij}, \tag{14.3}$$

where $\mathbf{X}$ is the matrix with the coordinates of the A- and the M-points. In the configurations of Figure 14.5, all between-sets distances are equal. Thus, $d_{ij}(\mathbf{X}) - \widehat{d}_{ij} = 0$, for all defined p_{ij}, but $\sum_{i<j} w_{ij} d_{ij}^2(\mathbf{X}) > 0$. This condition means that $\sigma_1(\mathbf{X})$ is zero, irrespective of the proximities. This degenerate solution is not limited to ordinal transformations. Even interval unfolding (with intercept one and slope zero) may lead to constant disparities yielding the trivial solutions above. The trivial solution of Figure 14.5a is a special case of the one discussed in Section 13.1. For ordinal or interval unfolding, it always exists, because the within-sets proximities are missing and the between-sets disparities all can be made equal.

Although these equal disparity solutions seem without any information about the data, Van Deun, Groenen, Heiser, Busing, and Delbeke (2005) showed that still a meaningful interpretation of such a solution is possible. The important idea is that one needs to zoom in on the points that are clustered together. Then it turns out that these points have different positions that depend on the data. The interpretation is done by projection using the so-called signed-compensatory distance model. For more information, we refer to Van Deun et al. (2005). Of course, without zooming, no useful information of the equal disparity solution can be derived.

To avoid these solutions, Kruskal (1968) and Kruskal and Carroll (1969) proposed the use of a variant of the stress measure called *Stress2* or *Stress-form2*,

$$\sigma_2(\mathbf{X}) = \left(\frac{\sum_{i<j} w_{ij}(d_{ij}(\mathbf{X}) - \widehat{d}_{ij})^2}{\sum_{i<j} w_{ij}(d_{ij}(\mathbf{X}) - \bar{d})^2} \right)^{1/2}, \text{ for all defined } p_{ij}, \tag{14.4}$$

where $\bar{d}$ denotes the mean of all distances over which the summation extends (see Section 11.2). For the above solutions where all between-sets distances are strictly equal, we find that $\sigma_2(\mathbf{X})$ is not defined because $\sum_{i<j} w_{ij}(d_{ij}(\mathbf{X}) - \bar{d})^2 = 0$, which leads to 0/0. However, if an MDS program is started from a configuration slightly different from the trivial solution, the program iterates away from the trivial solution. The reason is that close to the trivial solution σ_2 is large, because the denominator of (14.4) is close to zero. Several computer programs for ordinal MDS offer an option for minimizing σ_2 rather than σ_1. The criterion σ_2 always (except

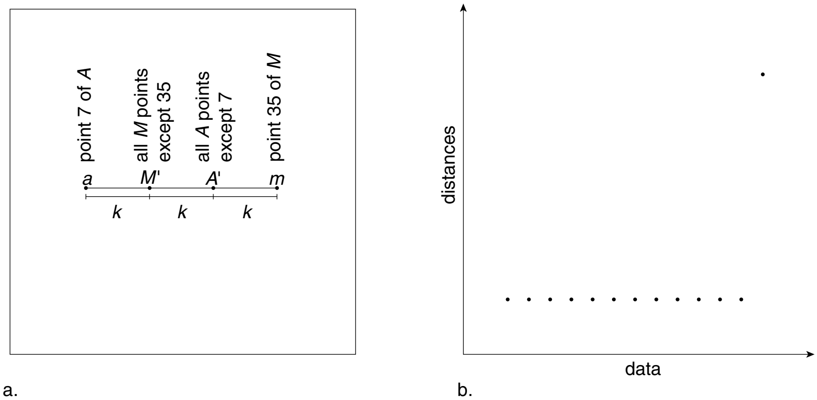

FIGURE 14.6. Trivial solution for ordinal unfolding under σ_2 (after Carroll, 1980).

at 0) yields values higher (typically twice as large) than σ_1, because it has the same numerator but a smaller denominator. Thus, using σ_2 avoids the equal-distance trivial solution.[5]

The Four-Point Solution

Using σ_2 does not free unfolding from degeneracies totally. If we compute the distances for the AM configuration in Figure 14.3 and use the between-sets distances as data for a 1D unfolding representation under σ_2, then the four-point configuration in Figure 14.6 is a perfect but trivial solution (Kruskal & Carroll, 1969; Carroll, 1980). It represents all A-points of Figure 14.3 by A', except for point 7, which corresponds to a. Similarly, all M-points of Figure 14.3 are mapped into M', except for point 35, which is carried into m. Because only the distances between A and M define the solution, σ_2 involves only two distance values, k and $3k$. $3k$ represents the greatest distance of the AM configuration, and k represents all other distances. Hence, the Shepard diagram essentially exhibits a horizontal array of points, except that the last point to the right is shifted upwards so that its value on the ordinate is three times that of the other points. This step function is perfectly monotonic, which makes the numerator of σ_2 equal to zero. At the same time, the norming factor $(d_{ij} - \bar{d})^2$ is not equal to zero. Therefore, $\sigma_2 = 0$.

This degeneracy is somewhat contrived and not likely to occur often, if at all, in real applications. It shows, however, that the norming factor used

[5] Although σ_2 tends to keep the *variance* of the distances large, this does not prevent degeneracies in "regular" ordinal MDS (see Section 13.1).

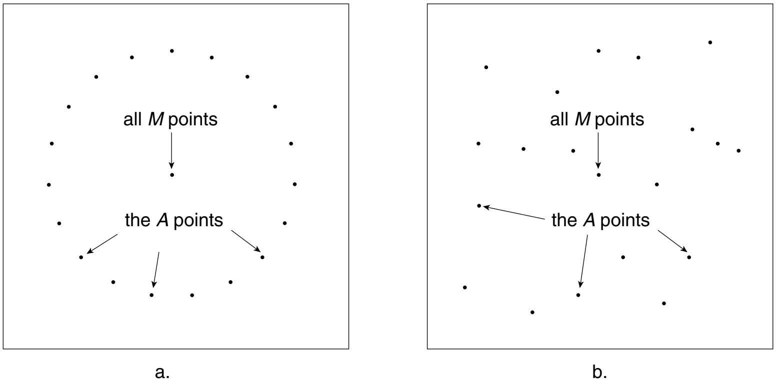

FIGURE 14.7. Trivial row-conditional ordinal unfolding solutions for the AM data (with the points of A in the rows) using Stress, panel a., and using (14.4), panel b.

in σ_2 has alleviated the degeneracy problem only to a degree. More specific degeneracies are discussed in Heiser (1989a).

Trivial Solutions for Row-Conditional Unfolding

For preference rank-orders, it is quite natural to have independent ordinal transformations for each of the individuals. If the individuals are represented by the rows, then it means that the data are treated row-conditionally. Again, minimizing Stress using a row-conditional transformation of at least interval level may lead to a zero Stress solution with equal distances as in Figure 14.5.

However, treating the transformations row-conditionally, also introduces additional trivial unfolding solutions. Consider the AM data, where the A points are the rows. Then, the equal distance solution in panel a. of Figure 14.7 looks similar to panel b. of Figure 14.5. The difference lies in the role of the points in the center which are the rows in Figure 14.5b and the column points (M) in Figure 14.7a.

A second trivial solution may occur when minimizing (14.4) with row-conditional transformations. In Figure 14.7b, all column points (M) are again represented in the center, but the row points scatter through the space. The row-conditional transformation has allowed different distances between row points to the cluster of column points in the center while keeping the distances within a row equal. In (14.4), the numerator is zero because all distances are the equal to the d-hats within each row. The denominator is nonzero because the distances from a row point to the cluster differ per row. This solution can be considered degenerate because it is independent of the data.

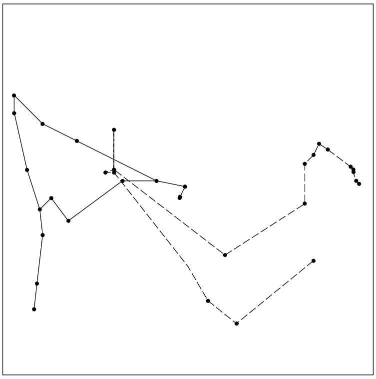

FIGURE 14.8. Row-conditional unfolding representation based on distances between points in A and M in Fig. 14.3.

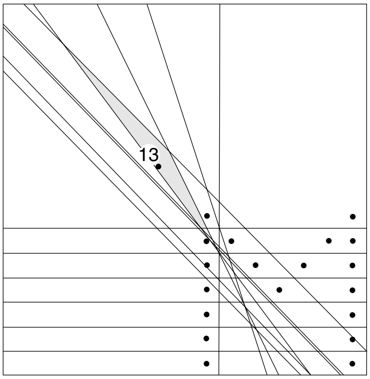

FIGURE 14.9. Isotonic region (shaded) for point 13 in configuration of Fig. 14.3; boundaries defined by order of distances of 13 to points in M.

14.5 Isotonic Regions and Indeterminacies

To get a feeling for the uniqueness or, expressed conversely, the indeterminacies of a conditional, ordinal unfolding representation we return again to our AM configuration in Figure 14.3 and use its distances. In conditional unfolding, there are two possible analyses: we may use the 16×19 proximity matrix in which A's points form the rows and M's points the columns, or the 19×16 transposed matrix in which the roles of A and M are reversed. We choose the first approach, which implies that only the distances from each point in A to every point in M are constrained by the data, but not the distances from each point in M to every point in A. (You may think of A as the set of ideal points and of M as the set of points representing choice objects.) The SSAR-2 program then leads to Figure 14.8, with the low alienation $K = 0.002$ [see (11.6)]). We note that there is a substantial deformation of the letters, in fact, a much stronger one than for the unconditional case. The M, in particular, can hardly be recognized. As could be expected, the row-conditional unfolding does not recover the underlying configuration nearly as well as the unconditional version.

In Figure 14.8, we can move the points around quite a bit without making the alienation worse. One example of what is possible is the underlying configuration itself (Figure 14.3), for which $K = 0$. Hence, the SSAR-2 solution is only weakly determined, that is, many more configurations exist with equal or even better fit to the data. This implies that it may be risky to embark on substantive interpretations of such representations, so we should study when we may do so.

A natural first question is whether the poor recovery of the AM configuration is a consequence of certain properties that are not likely to hold in

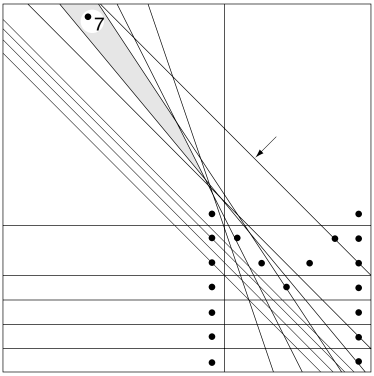

FIGURE 14.10. Isotonic region (shaded) for point 7, defined as in Fig. 14.9.

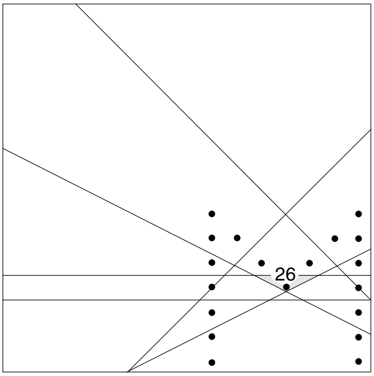

FIGURE 14.11. Isotonic region (shaded) for point 26, defined by its distances to all other points.

general. To answer this question, let us check the invariance of some of the points. Assume that M is fixed and that A's points have to be located under the constraints of conditional unfolding. For point 13, which is closest to M, we obtain as its solution space or *isotonic region* (i.e., the region in which the distances of every point to M's points are ordered equivalently) the grey area shown in Figure 14.9. Note that all boundaries are straight lines in the conditional case, in contrast to the unconditional MDS considered in Chapter 2. The indeterminacy of point 13 is considerable but not unlimited.

Determining the isotonic region for point 7 in a similar fashion leads to Figure 14.10. We notice immediately that this point's solution space is much greater and is closed to the outside only by the boundary line marked with the arrow. Thus, point 7 could be positioned much farther to the outside of this region without affecting the fit of the conditional unfolding solution at all. But why is this point's solution space so much greater than the one for point 13? One conjecture is that the boundary lines for those points that are closer to the M differ more in their directions, which leads to a network with tighter meshes. To test this conjecture, we look at the isotonic region of point 26 relative to all other points in M. Figure 14.11 shows that the boundary lines indeed run in many very different directions, which generates a comparatively small isotonic region, even though many fewer order relations are involved than in the above. The number of constraints as such does not imply anything about the metric determinacy of a point. What is important is how ideal and object points are distributed throughout the space relative to each other.

The best relative distribution of ideal points and object points is one where they are thoroughly mixed, that is, where both are evenly spread

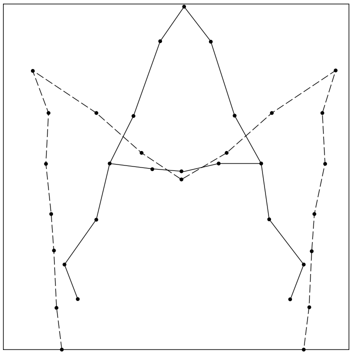

FIGURE 14.12. Row-conditional unfolding representation of distances from superimposed A and M point sets.

throughout the space. Substantively, this implies that we have individuals with many different preference patterns, so each object is someone's first choice. With our AM configuration, a situation like this can be approximated by superimposing A on M, which is done here by shifting A and M so that their respective centroids coincide with the origin. With the distances of this configuration, SSAR-2 leads to Figure 14.12, with $K = 0.002$. The metric recovery of the underlying configuration is virtually perfect, as expected. Thus, conditional unfolding does work—under favorable circumstances!

Part of the favorable circumstances of the situation leading to Figure 14.12 was also that the number of ideal and object points was high for a 2D solution. Coombs (1964) has shown that if there are n object points in an $(n-1)$-dimensional MDS space, then *all* isotonic regions for the ideal points are open to the outside. Why this is so is easy to see for the special case of three object points in the plane. We connect the points A, B, and C by straight-line segments, and draw straight lines running perpendicularly through the midpoints of the line segments. These lines will then intersect at just one point, which is the center of the circle on which A, B, and C fall. Moreover, the three lines will partition the plane into six regions, which are all open to the outside. Any ideal point falls into one of these regions, depending on the empirical preference order for the individual it represents. With three objects, there are exactly six different rank-orders, corresponding to the six regions. But, because all regions are open, the location of the ideal points is very weakly determined indeed. If the number of object points grows relative to the dimensionality of the representation space, then more and more closed regions result. These regions are located primarily where the object points are, as we concluded above.

TABLE 14.2. Similarity data for breweries A, ..., I and attributes 1, ..., 26.

	A	B	C	D	E	F	G	H	I
1	3.51	4.43	4.76	3.68	4.77	4.74	3.43	5.05	4.20
2	3.41	4.05	3.42	3.78	1.04	3.37	3.47	3.25	3.79
3	3.20	3.66	4.22	3.07	3.86	4.50	3.19	4.62	3.75
4	2.73	5.25	2.44	2.75	5.28	2.11	2.68	2.07	3.63
5	2.35	3.88	4.18	2.78	3.86	4.37	2.38	4.21	4.63
6	3.03	4.23	2.47	3.12	4.24	2.47	2.90	2.36	3.53
7	2.21	3.27	3.67	2.49	3.40	4.10	2.53	4.03	3.33
8	3.91	2.71	4.59	3.91	4.23	4.72	3.81	4.88	3.96
9	3.07	4.08	4.74	3.34	4.23	4.88	3.20	5.20	3.95
10	3.21	3.57	4.20	3.24	3.85	4.28	3.16	4.30	3.75
11	3.15	3.80	4.34	3.33	3.88	4.49	3.17	4.70	3.67
12	2.84	3.41	4.01	2.89	3.64	4.15	2.95	4.25	3.65
13	2.75	3.24	4.07	2.68	3.55	4.18	2.84	4.56	3.22
14	2.35	3.44	4.13	3.16	3.55	4.55	2.82	4.49	3.29
15	3.07	3.82	4.17	3.21	3.94	4.42	3.21	4.41	3.67
16	3.45	4.29	4.44	3.74	4.47	4.68	3.61	4.76	4.04
17	2.53	4.71	4.53	2.83	4.83	4.71	2.70	4.83	4.72
18	3.12	3.58	4.10	3.14	3.82	4.28	3.10	4.53	3.50
19	2.93	3.27	4.13	2.80	3.46	4.10	2.84	5.12	3.13
20	2.24	3.11	4.12	2.39	3.39	4.17	2.54	4.33	3.19
21	2.41	3.14	3.43	2.40	3.22	3.45	2.43	3.22	3.93
22	3.32	3.74	4.32	3.32	4.01	4.64	3.26	4.88	3.72
23	3.39	4.04	4.51	3.48	4.23	4.63	3.43	4.95	3.86
24	2.88	3.39	3.85	2.90	3.61	4.18	2.79	3.94	3.96
25	2.74	3.57	2.37	2.77	3.96	2.49	2.71	2.44	3.26
26	2.70	3.10	3.85	2.82	3.58	4.13	2.79	4.17	3.20

14.6 Unfolding Degeneracies in Practice and Metric Unfolding

We now demonstrate some of the degeneration problems with the data in Table 14.2. Beer drinkers were asked to rate nine breweries on 26 attributes (Borg & Bergermaier, 1982). The attributes were, for example, "Brewery has rich tradition" or "Brewery makes very good Pils beer". Relative to each attribute, the informant had to assign each brewery a score on a 6-point scale ranging from 1 = not true at all to 6 = very true. The resulting scores are therefore taken as similarity values.

Minimizing Stress (σ_1) in unconditional ordinal unfolding, KYST yields a computer printout similar to Figure 14.13a. We find that all of the brewery points are tightly clustered, whereas all of the attribute points lie on a J-shaped curve. The Shepard diagram for this configuration is given in Figure 14.13b. At first sight, these results do not look degenerate, even though the extremely low Stress of $\sigma_1 = .0005$ would at least suggest this possibility. Indeed, a second look at the Shepard diagram reveals that the distances scatter over only a small range. Thus, they are very similar, in spite of the considerable scatter in the diagram. The horizontal step function in Figure 14.13b is the monotone regression line. So, the sum of the squared (vertical) distances of each point from this line defines the numerator of Stress, which is definitely much smaller than the sum of the squared distance coordinates

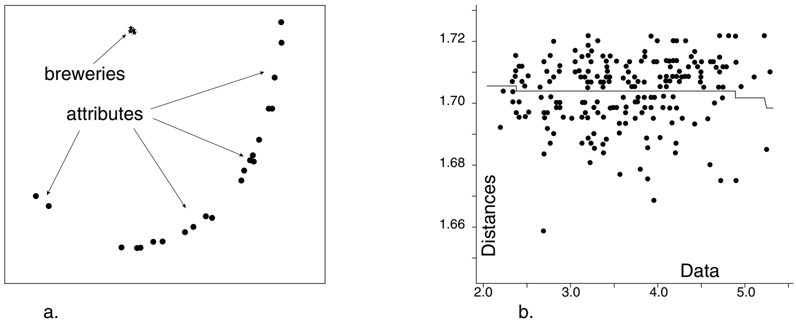

FIGURE 14.13. Ordinal unfolding representation (a) of data in Table 14.2, using Stress, σ_1, and (b) its Shepard diagram.

of the points in the Shepard diagram, the denominator of Stress. The J-shaped curve in Figure 14.13a thus turns out to be a segment of a circle with its origin at the brewery points. Thus, this example is a degenerate solution of the equal distance type shown in Figure 14.5.

Instead of ordinal unfolding, stronger assumptions (or hypotheses) about the data can be imposed, because metric MDS is often more robust than ordinal MDS. If it seems justifiable to assume that the proximities are at least roughly interval scaled, using metric MDS is no problem. But even if this is not the case, one could replace the original data with appropriate ranking numbers and then use interval MDS, because the rank-linear model is very robust vis-à-vis nonlinearities in the relations of data and distances, as we saw in Chapter 3. For metric conditional unfolding, we have

$$p_{ij} \mapsto a_i + b_i \cdot p_{ij} \approx d_{ij}, \tag{14.5}$$

where i denotes an individual, j is an object, and $\approx$ means as nearly equal as possible. In the unconditional case, the intercept a and slope b are equal for every individual i; that is,

$$p_{ij} \mapsto a + b \cdot p_{ij} \approx d_{ij}. \tag{14.6}$$

Using (unconditional) interval unfolding, however, has little effect for the data in Table 14.2 and leads to virtually the same configuration as in Figure 14.13a. Moreover, it has the additional drawback that now the regression line in the Shepard diagram has the "wrong" slope: given that the data are similarities, the regression line should run from the upper left-hand corner to the lower right-hand corner of the diagram in order to preserve the interpretation of the individuals' points as ideal points or, in the present case, the direct correspondence of geometrical and psychological closeness.

We see that using Stress as a minimization criterion can lead to wrong solutions. This is easy to see because the configuration in Figure 14.13a suggests that all breweries are evaluated in the same way with respect to

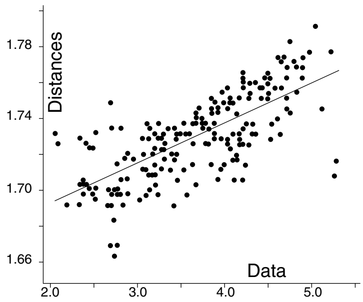

FIGURE 14.14. Shepard diagram of linear unfolding of data in Table 14.2 using Stress, σ_1.

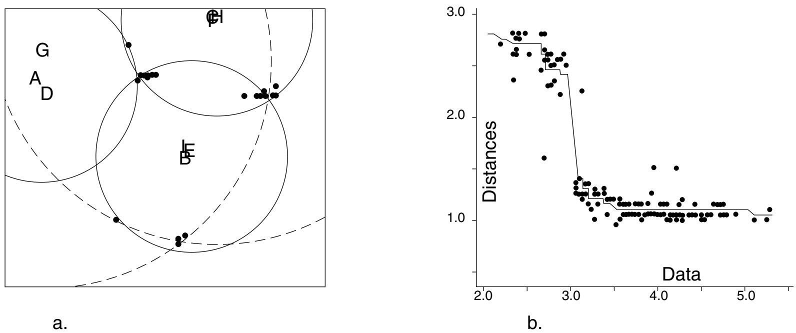

FIGURE 14.15. Ordinal unfolding representation (a) of data in Table 14.2, using σ_2, and (b) its Shepard diagram.

all attributes. From the empirical data in Table 14.2, this cannot be true. When we use σ_2, it becomes far more difficult to diagnose, from looking at the configuration, that something went wrong. The ordinal unfolding solution (under σ_2) is shown in Figure 14.15a. The letters A, ..., I stand for the nine breweries, the solid points for the 26 attributes. The figure suggests that the breweries form three groups, and the attributes also seem to cluster to some extent. But the Shepard diagram for the unfolding solution (Figure 14.15b) shows immediately that we have a degeneracy of the two-distance-classes type. Although the data scatter quite evenly over the range 2.0 to 5.5, there are practically only two distances. All of the small proximities up to about 3.0 are mapped into distances of about 2.5, whereas all other proximities are represented by distances about equal to 1.2. Almost all points lie very close to the regression line; thus, σ_2 is very low.

After learning from the Shepard diagram that there are essentially only two different distances in the scaling solution, we can identify them. Because we are only concerned with between-sets distances, we have to show that each distance from a brewery point to an attribute point is equal to

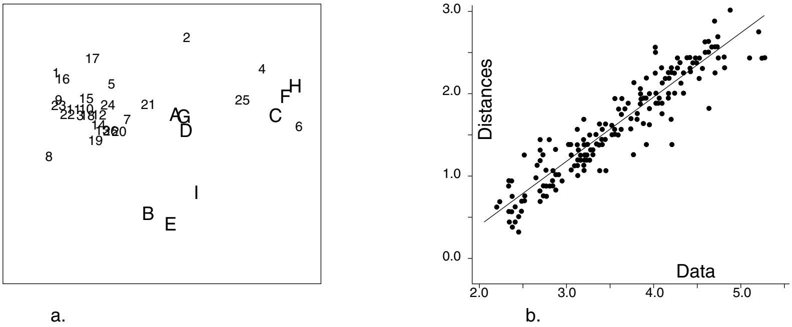

FIGURE 14.16. Linear unfolding representation (a) of data in Table 14.2, using σ_2, and its Shepard diagram (b).

either a or b, where $a < b$. Moreover, because the unfolding was done unconditionally, the same would be true in the reverse direction, that is, from each attribute point to all brewery points. In Figure 14.15a, the two distance types are indicated (for the perspective from the brewery points to the attribute points) by either solid circles (for a-type distances) or broken circles (for b-type distances). Similar circles, with radius equal to either a or b, could be drawn about the attribute points in such a way that the brewery points would fall onto or close to them.

As we did for Stress, we now unfold the data with an interval regression approach. The solution is given in Figure 14.16a, where the brewery points are labeled A, ..., I , as above, and the attribute points as 1, ..., 26. The brewery points tend to arrange themselves in the same groups as in the degenerate solution in Figure 14.15a for empirical reasons, as the Shepard diagram in Figure 14.16b shows. The distances and the proximities of the unfolding solution vary over a wide range. There are no gaps in the distribution, and the linear regression line fits very well. The problem with this solution is that the slope of the regression line is not as we would like it to be. If this is not noticed by the user, serious interpretational mistakes are bound to result. The configuration in Figure 14.16a puts a brewery closer to an attribute the less (!) this brewery was judged to possess this attribute. Thus, for example, brewery A is not really close to attribute 21 as the configuration suggests; rather, the contrary is true. This certainly leads to an awkward and unnatural meaning for the configuration, where two points are close when the objects they represent are psychologically different.

We conclude that using σ_2 instead of σ_1 does not eliminate the problems of unfolding. In the ordinal case, we again get a degenerate solution (even though it is somewhat less degenerate than for σ_1). For the metric approach, we obtain an undesirable inverse representation that is hard to interpret.

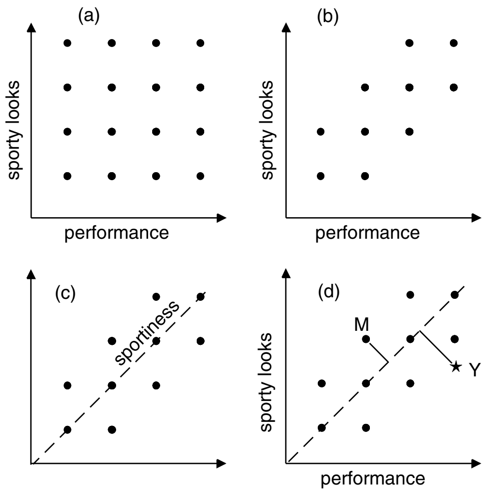

FIGURE 14.17. Hypothetical example to demonstrate problems of dimensional interpretations in unfolding.

14.7 Dimensions in Multidimensional Unfolding

Apart from degeneracies and indeterminacies, there are further problems in unfolding that one should be aware of when interpreting an unfolding solution. Consider an example. Assume that we want to know how an individual selects a car from a set of different automobiles. Assume further that the preference judgments were made in a 2D unfolding space with dimensions "performance" and "sporty looks". Figure 14.17a shows 16 hypothetical cars in a space spanned by these dimensions. A market researcher wants to infer this space from the person's similarity data. This is a difficult task if, as Figure 14.17b illustrates, there are no cars in the upper left- and the lower right-hand corners. The reason for the empty corners in this example is that cars with a very high performance must look sporty to some extent, for engineering reasons. The converse is usually also true empirically; that is, cars with extremely poor performance do not look like racing machines. But with the remaining 10 cars it is likely that the researcher would conclude that essentially only one dimension explains the similarity data, especially because the resulting dimension ("sportiness") seems to make sense psychologically (Figure 14.17c).

Figure 14.17d shows the consequences of this false interpretation. Let Y be the ideal point of some individual. This individual wants a car with very high performance and moderately sporty looks. A market researcher, therefore, should recommend making car M in Figure 14.17d less sporty in looks and more powerful in its performance. However, on the basis of the accepted unfolding solution, the market researcher would come to a different, incorrect conclusion: with “sportiness” as the assumed decision criterion, the advice would be to increase M’s sportiness so that M would move closer to Y on this dimension. Concretely, this movement could be achieved in two ways: increase performance and/or sporty looks. Because the latter is cheaper and easier to implement, this would be the likely immediate action. But this would be just the wrong thing to do, because the person wanted a reduction, not an increase, in the sporty looks of M.

The problems encountered here are a consequence of the fact that some corners of the similarity space remain empty. Coombs and Avrunin (1977, p. 617) therefore argue that “deliberate efforts” should be made to avoid collapsing the preference space due to correlated dimensions. This means, in practice, that an unfolding analysis should be based on a set of objects that are carefully selected from the product space of the presumed choice criteria, not on a haphazard collection of objects.

14.8 Multiple Versus Multidimensional Unfolding

When aggregated data are analyzed in MDS, there is always a danger that the multidimensionality is an aggregation artifact. This danger is particularly acute in multidimensional unfolding because here the data are usually from different individuals.

Unfolding assumes that all individuals perceive the world in essentially the same way. There is just one configuration of objects. Differences among individuals are restricted to different ideal points. If this assumption is not correct, unfolding preference data will be misleading. Consider an example.

Norpoth (1979a) reports two data sets, where German voters were asked to rank-order five political parties in terms of preference. The parties ranged from Nationalists to Communists, and so one could expect that the respondents should have agreed, more or less, on the position of each party on a left-to-right continuum.

Running an unfolding analysis on these data, Norpoth (1979a) concluded that he needed a 2D solution for an adequate representation of the data. The solution shows one dimension where the Communists are on one end and the Nationalists are on the other. This is interpreted as the familiar left-to-right spectrum. The second dimension shows the (then) ruling coalition parties on one end and the major opposition party on the other. This interpretation also seemed to make sense.

One can question, however, whether all voters really perceived the parties in the same way. One hypothesis is that the voters do indeed all order the parties on a left-to-right dimension, but that they do not always agree on where these parties are located relative to each other. Indeed, Van Schuur (1989) and Borg and Staufenbiel (1993) independently showed for Norpoth’s data that by splitting the set of respondents into two groups (in each sample) by simply placing the Liberals to the right of the Conservatives in one case, and to the left of the Conservatives in the other, while leaving all other parties ordered in the same way, two subsamples are obtained that each yield one-dimensional unfolding solutions.

Substantively, such multiple solutions are much more convincing: they preserve a simple dimensional model of how political parties are perceived; they explain different preferences by a simple ideal-point model; and, finally, they account for group differences by a simple shift of the position of the Liberals, an ambiguous party in any case.

There exist computer programs for multiple one-dimensional unfolding (e.g., Lingoes, 1989; Van Schuur & Post, 1990). They offer the easiest way to test for the existence of multiple 1D unfolding scales.

14.9 Concluding Remarks

Unfolding is a natural extension of MDS for two-way dissimilarity data. When no transformation is allowed on the data (or a ratio transformation), unfolding can be safely used. However, if transformations are required, for example, for preference rank-orders, then special caution is needed because the usual approaches yield a degenerate solution with all disparities being equal. Chapter 15 discusses several of such solutions.

14.10 Exercises

Exercise 14.1 Consider the unfolding solution for the breakfast items in Figure 14.2. Attempt an interpretation. In particular, find "labels" for the four groups of breakfast items, and interpret their positions relative to each other. (What lies opposite each other, and why?)

Exercise 14.2 Consider the (contrived) color preferences of six persons (A..F) in the table below (Davison, 1983). The data are ranks, where 1 = most preferred.

	Person					
Color	A	B	C	D	E	F
Orange	1	2	3	4	3	2
Red	2	1	2	3	4	3
Violet	3	2	1	2	3	4
Blue	4	3	2	1	2	3
Green	3	4	3	2	1	2
Yellow	2	3	4	3	2	1

(a) Unfold these data without any transformations.

(b) Discuss the solution(s) substantively, relating them to Figure 4.1 and to unfolding theory. In what sense are the six persons similar, in what sense do they differ?

(c) Discuss technical reasons why the unfolding analysis works for these data.

(d) Construct a set of plausible color preference data that do not satisfy the ideal point model.

(e) Discuss some data sets that satisfy the ideal point model but that would most likely lead to degenerate or other nondesirable MDS solutions. (Hint: Consider the distribution of ideal points in the perceptual space.)

Exercise 14.3 The following table shows empirical color preferences of 15 persons (Wilkinson, 1996). The data are ranks, where 1 = most preferred.

	Person														
Color	A	B	C	D	E	F	G	H	I	J	L	M	N	O	P
Red	3	1	3	1	5	3	3	2	4	2	1	1	1	2	1
Orange	5	4	5	3	3	2	4	4	5	5	5	5	4	5	2
Yellow	4	3	1	5	2	5	5	3	3	4	2	4	5	3	3
Green	1	5	4	4	4	1	2	5	1	3	4	2	2	4	4
Blue	2	2	2	2	1	4	1	1	2	1	3	3	3	1	5

(a) Unfold these data.

(b) Discuss the solution(s) substantively, connecting the color points in the order of the electromagnetic wavelengths of the respective colors.

(c) Use an external starting configuration where the color points are positioned on a rough color circle similar to the one in Figure 4.1. (Hint: Place the person points close to their most preferred color points in the starting configuration.)

(d) Compare the unfolding solutions with and without external starting configurations, both technically in terms of Stress and substantively in terms of a reasonable theory.

Exercise 14.4 The following table shows the dominant preference profiles (columns) for German political parties in 1969. A score of 1 indicates "most preferred". The row "freq" shows the frequency of the respective preference order in a representative survey of 907 persons (Norpoth, 1979b).

	Preference Type										
Political Party	1	2	3	4	5	6	7	8	9	10	11
SPD (Social Democrats)	1	1	1	1	3	3	2	2	2	2	3
FDP (Liberals)	2	2	3	3	2	2	3	3	4	1	1
CDU (Conservatives)	3	3	2	2	1	1	1	1	1	3	2
NPD (Nationalists)	4	5	4	5	5	4	5	4	3	5	4
DKP (Communists)	5	4	5	4	4	5	4	5	5	4	5
Freq	29	85	122	141	56	66	135	138	11	16	19

(a) Unfold these data in one to three dimensions and discuss the solutions. Use both ordinal and linear MDS, and both unweighted and weighted (by "freq") unfolding.

(b) Norpoth (1979a) claims that these data require a 2D unfolding space. Yet, most Germans would probably order these parties from left to right as DKP-SPD-FDP-CDU-NPD or as DKP-SPD-CDU-FDP-NPD. Sketch diagrams for these two orders, where the Y-axis represents preference ranking—the highest rank 1 getting the highest Y-score—and the X-axis the left-to-right order. What do these diagrams show you with respect to single-peakedness of the preference functions? Can you accommodate most preference profiles in the scales? Can you accommodate them in one single scale too?

(c) Compute two (or more) 1D unfoldings for subsets of the voter profiles as an alternative to one common unfolding solution for all persons combined. Discuss the substantive implications.