16
Special Unfolding Models

In this chapter, some special unfolding models are discussed. First, we distinguish internal and external unfolding. In the latter case, one first derives an MDS configuration of the choice objects from proximity data and afterwards inserts ideal points to represent preference data. Then, the vector model for unfolding is introduced as a special case of the ideal-point model. In the vector model, individuals are represented by vectors and choice objects as points such that the projections of the objects on an individual's vector correspond to his or her preference scores. Then, in weighted unfolding, dimensional weights are chosen freely for each individual. A closer investigation reveals that these weights must be positive to yield a sensible model. A variant of metric unfolding is discussed that builds on the Bradley–Terry–Luce (BTL) choice theory.

16.1 External Unfolding

We now turn to *external unfolding* models. These models assume that a similarity configuration of the choice objects is given, possibly obtained from a previous MDS analysis. If we have preference data on these objects for one or more individuals, then external unfolding puts a point (*ideal point*) for each individual in this space so that the closer this point lies to a point that represents a choice object, the more this object is preferred by this individual. In an *internal unfolding* problem, by contrast, only the preference data are given, from which both the object configuration and the ideal points have to be derived. Thus, external unfolding for the breakfast objects, say, would require a coordinate matrix on the objects A, ..., O and, in addition, preference data as in Table 14.1. The coordinate matrix could be obtained from an MDS analysis of an additional matrix of proximities for the 15 objects. Afterwards, an ideal point S would have to be embedded into this MDS configuration for each person in turn such that the distances from S to the points A, ..., O have an optimal monotonic correspondence to the preference ranks in Table 14.1.

Finding the optimal location for individual i's ideal point is straightforward. Consider the majorization algorithm for internal unfolding in Section 14.2. The coordinates for the set of objects, $\mathbf{X}_1$, are given and hence are fixed. Thus, we only have to compute iteratively the update for $\mathbf{X}_2$, the coordinates of the individuals, given by (14.2). Instead of δ_{ij} we may use $\widehat{d}_{ij}$ to allow for admissibly transformed preference values of individual i with respect to object j. In this case, we do not have to be concerned about degenerate solutions, because the coordinates of the objects are fixed. Because the distances among the individuals do not represent any data (the within-individuals proximities are missing in external unfolding), the individuals' points can be computed one at a time or simultaneously without giving different solutions. However, if the coordinates of the individuals are fixed and we use external unfolding to determine the coordinates of the objects, then the trivial solution in Figure 14.7b can occur in which all objects collapse in one point.

In Figure 14.1, we saw that unfolding can be viewed as MDS of two sets of points (represented by the coordinates in $\mathbf{X}_1$ for the individuals and $\mathbf{X}_2$ for the objects), where the within-sets proximities are missing. Additionally, in external unfolding, $\mathbf{X}_1$ (or $\mathbf{X}_2$) is fixed. Groenen (1993) elaborates on this idea to identify special cases for MDS on two sets of objects. Table 16.1 shows some relations of the MDS models. For example, if the proximity weights w_{ij} of the within-individuals and within-objects proximities are nonmissing ($\mathbf{W}_{11} \neq \mathbf{0}$, $\mathbf{W}_{22} \neq \mathbf{0}$), and all coordinates of $\mathbf{X}_1$ and $\mathbf{X}_2$ are free, then the model is full MDS. But if $\mathbf{W}_{11} = \mathbf{0}$ and $\mathbf{W}_{22} = \mathbf{0}$, we have (internal) unfolding. For *almost complete* MDS we have one of the within-blocks portion of the data matrix missing. (Therefore, almost complete MDS appears twice in Table 16.1.) It is *semi-complete* MDS if additionally $\mathbf{X}_1$ is fixed, so that the between-blocks and within-objects proximities are fitted by $\mathbf{X}_2$ for given $\mathbf{X}_1$. Note that for fixed $\mathbf{X}_1$, $\mathbf{W}_{11}$ is immaterial and $\mathbf{W}_{22}$ determines the model.

16.2 The Vector Model of Unfolding

The ideal-point model for unfolding has a popular companion, the *vector model* of unfolding, which goes back to Tucker (1960). It differs from the

TABLE 16.1. Relation of unfolding, external unfolding, and MDS using the partitioning in two sets, $\mathbf{X}_1$ (objects) $\mathbf{X}_2$ (individuals), as in Figure 14.1. We assume that $\mathbf{X}_2$ is always free and $\mathbf{W}_{12} \neq \mathbf{0}$.

			Model
$\mathbf{X}_1$ Free	$\mathbf{W}_{11} = \mathbf{0}$	$\mathbf{W}_{22} = \mathbf{0}$	Unfolding
$\mathbf{X}_1$ Free	$\mathbf{W}_{11} = \mathbf{0}$	$\mathbf{W}_{22} \neq \mathbf{0}$	Almost complete MDS
$\mathbf{X}_1$ Free	$\mathbf{W}_{11} \neq \mathbf{0}$	$\mathbf{W}_{22} = \mathbf{0}$	Almost complete MDS
$\mathbf{X}_1$ Free	$\mathbf{W}_{11} \neq \mathbf{0}$	$\mathbf{W}_{22} \neq \mathbf{0}$	Complete MDS
$\mathbf{X}_1$ Fixed	$\mathbf{W}_{11} = \mathbf{0}$	$\mathbf{W}_{22} = \mathbf{0}$	External unfolding
$\mathbf{X}_1$ Fixed	$\mathbf{W}_{11} = \mathbf{0}$	$\mathbf{W}_{22} \neq \mathbf{0}$	Semi-complete MDS
$\mathbf{X}_1$ Fixed	$\mathbf{W}_{11} \neq \mathbf{0}$	$\mathbf{W}_{22} = \mathbf{0}$	External unfolding
$\mathbf{X}_1$ Fixed	$\mathbf{W}_{11} \neq \mathbf{0}$	$\mathbf{W}_{22} \neq \mathbf{0}$	Semi-complete MDS

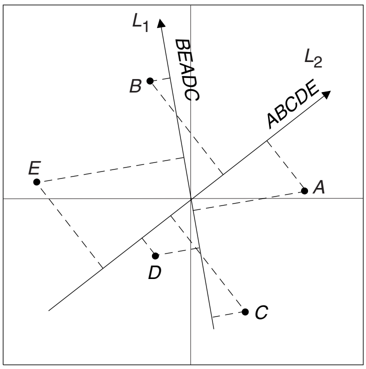

FIGURE 16.1. Illustration of the vector model of unfolding; L_1 and L_2 represent two individuals.

ideal-point model in representing each individual i not by a point but by a directed line (segment), a vector. From now on, we switch to the notation $\mathbf{X}$ for the objects and $\mathbf{Y}$ for the individuals.

Representing Individuals by Preference Vectors

For each individual i, a linear combination of the coordinate vectors of $\mathbf{X}$ is to be found so that it corresponds as much as possible to the preference data $\mathbf{p}_i$ of this individual. Figure 16.1 should clarify the situation. The diagram shows a configuration of five choice objects (points $A, \ldots, E$) and, in addition, two preference lines, L_1 and L_2.

Assume that individual i had ordered the objects as $A > B > C > D > E$ in terms of preference. Then, L_2 is a perfect (ordinal) representation of i's preferences, because the projections of the object points onto this line perfectly match i's preference rank-order.

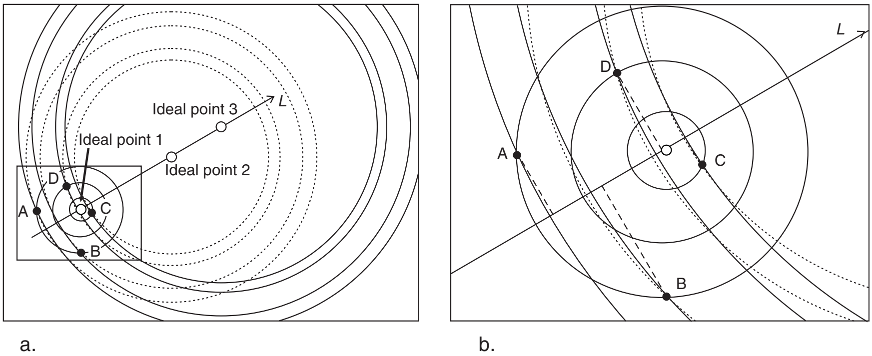

FIGURE 16.2. Illustrating the relation of the ideal-point and the vector model for preferential choice. Panel (a) shows that the more the ideal points move away from the object points (solid points) on line L, the more the ideal-point model approximates the vector model. Panel (b) zooms in on the box in panel (a). The straight dashed lines show the projections according to the vector model.

Of course, with so few points, many other lines around L_2 would be equally perfect representations for the empirical preference order $A > \cdots > E$. On the other hand, any arbitrary direction would not do, and for some preference orders (such as $D > C > E > A > B$) no perfect representation exists at all in Figure 16.1.

The vector model is, in a way, but a special case of the ideal-point model. To see this, consider Figure 16.2 and assume that person i's ideal point I is moved from ideal point 1 to 3 along the direction of vector L. As I moves away from the object points $A, \ldots, D$ in the direction of line L, the iso-preference circles grow and grow in diameter, so that the circle segments from the object points to L will become increasingly less curved. When I's distance from the centroid of the object points approaches ∞, the circle segments approximate the straight projection lines of $A, \ldots, D$ onto L (Carroll, 1972; Coombs, 1975). Expressed differently, the distances from I to the various object points approximate the distances of I to the projections of the object points onto the line L. Hence, in terms of fitting the models to data, the ideal and the vector models become very similar. However, this does not imply that the psychological models also become equivalent (Van Deun, Groenen, & Delbeke, 2005). The main difference is that in the vector model, preferences are confined to a subspace of the unfolding space (i.e., the preference vector) and any variation in the surrounding space is ignored. In a dimensional interpretation of the unfolding models, we note that in the vector model the attributes (dimensions) contribute with fixed weights to the preference function of an individual, however close or distant the object points are from line L, whereas in the ideal-point model a low score on one attribute can be compensated by a very high score on other dimensions to lead to the same projection onto L (see also Section 14.7).

Hence, the vector model and the ideal-point model imply similar decision functions only for points that are close to the vector.

Apart from this difference, the vector model also represents a particular preference notion that can be described as "the more, the better" on all dimensions. Obviously, this property does not hold in general. For example, suppose that respondents have to rate how much they like teas of various temperatures. It is certainly not true that the hotter the tea the better. The opposite (the colder, the better) is not plausible either, not even for iced tea.

Metric and Ordinal Vector Models

In a metric model, the indeterminacy of locating L_i is eliminated or at least reduced, because the distances of the projection points on L_i are also meaningful in some quantitative sense. For example, if we require that $d(B, E) = d(E, D)$ on L_i, then only a line corresponding closely to the vertical coordinate axis may be selected as a representation. But then we could conclude that this individual based his or her preference judgments on the vertical dimension only, whereas some other person whose preference line is the bisector from the lower left-hand to the upper right-hand corner used both dimensions with equal weight. Note that if we put the arrowhead at the other end of the line, the person represented by this line would still weight both dimensions equally, but now the negative, not the positive, ends of each dimension are most attractive.

Fitting the Vector Model Metrically

In the vector model, one has to find an m-dimensional space that contains two sets of elements: (a) a configuration $\mathbf{X}$ of n points that represent the objects and (b) an m-dimensional configuration $\mathbf{Y}$ of N vectors that represent the individuals. The projections of all object points onto each vector of $\mathbf{Y}$ should correspond to the given preference data in the N columns of $\mathbf{P}_{n \times N}$. The model attempts to explain individual differences of preference by different weightings of the objects' dimensions.

Formally, we have the loss function

$$L(\mathbf{X}; \mathbf{Y}) = ||\mathbf{X}_{n \times m}\mathbf{Y}'_{m \times N} - \mathbf{P}_{n \times N}||^2. \qquad (16.1)$$

Note that $\mathbf{P}$ corresponds to the upper corner matrix in Figure 14.1. The vector model is fitted by minimizing (16.1) over $\mathbf{X}$ and $\mathbf{Y}$.

The loss function can be minimized by a singular value decomposition. Let $\mathbf{P} = \mathbf{K}\boldsymbol{\Lambda}\mathbf{L}'$ be the SVD of $\mathbf{P}$. Then, the first m columns of $\mathbf{K}\boldsymbol{\Lambda}$ and of

$\mathbf{L}$ define optimal solutions for $\mathbf{X}$ and for $\mathbf{Y}$, respectively. Setting $\mathbf{X} = \mathbf{K}$ and $\mathbf{Y} = \mathbf{L\Lambda}$ would do equally well.[1]

However, there are many more than just these two solutions. Minimizing $L(\mathbf{X};\mathbf{Y})$ by choice of $\mathbf{X}$ and $\mathbf{Y}$ does not uniquely determine particular matrices $\mathbf{X}$ and $\mathbf{Y}$. Rather, if $\mathbf{X}$ is transformed into $\mathbf{X}^* = \mathbf{XM}$ by a nonsingular matrix $\mathbf{M}$, then we simply have to transform $\mathbf{Y}$ into $\mathbf{Y}^* = \mathbf{Y}(\mathbf{M}^{-1})'$ to obtain the same matrix product. Such transformations can be conceived of as rotations and stretchings along the dimensions, because $\mathbf{M}$ can be decomposed by SVD into $\mathbf{P\Phi Q}'$, where $\mathbf{P}$ and $\mathbf{Q}$ are orthonormal and $\mathbf{\Phi}$ is a diagonal matrix of dimension weights [see (7.14)]. Geometrically, this means, for example, that one can stretch out a planar $\mathbf{X}$ along the Y-axis (like a rubber sheet), provided $\mathbf{Y}$ is stretched out along by the same amount along the X-axis. This destroys relations of incidence, for example, and thus makes interpretation difficult.

By restricting the vectors of $\mathbf{Y}$ to the same length (1, say), the model becomes more meaningful:

$$\begin{aligned} L(\mathbf{X};\mathbf{Y}) &= ||\mathbf{XY}' - \mathbf{P}||^2, \\ \mathrm{diag}(\mathbf{YY}') &= \mathrm{diag}(\mathbf{I}). \end{aligned} \tag{16.2}$$

The indeterminacy now reduces to a rotation; that is, $\mathbf{M}$ must satisfy $\mathbf{MM}' = \mathbf{I}$, because only then does $\mathbf{Y}^* = \mathbf{Y}(\mathbf{M}^{-1})'$ satisfy the additional side constraint in formula (16.2). This rotation is unproblematic for interpretations because it affects both $\mathbf{X}$ and $\mathbf{Y}$ in the same way because $\mathbf{Y}(\mathbf{M}^{-1})' = \mathbf{YM}$ if $\mathbf{M}$ is orthonormal.

Chang and Carroll (1969) developed a popular program, MDPREF, for solving the length-restricted vector model in (16.2). It first finds an SVD of $\mathbf{P}$ and then imposes the side constraint of unit length onto $\mathbf{Y}$'s vectors. Schönemann and Borg (1983) showed that this sequential approach may be misleading. The argument is based on first deriving a direct solution for (16.2). It exists only if the data satisfy certain conditions implied by the side condition $\mathrm{diag}(\mathbf{YY}') = \mathrm{diag}(\mathbf{I})$. Hence, (16.2) is a testable model that may or may not hold, whereas MDPREF always provides a solution.

If $\mathbf{X}$ is given, then things become very simple. The vector model for external unfolding only has to minimize (16.1) over the weights $\mathbf{Y}$. This problem is formally equivalent to one considered in Chapter 4, where we wanted to fit an external scale into an MDS configuration. If the preferences are rank-orders, then an optimal transformation also has to be computed.[2]

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[1] In contrast to ordinary PCA, $\mathbf{P}$ has individuals as column entries and the objects as row entries. Hence, the vector model for unfolding is sometimes referred to as a "transposed PCA."

[2] This model can be fitted by the PREFMAP program (for computational details, see Carroll, 1972).

Fitting the Vector Model Ordinally

Now, suppose that $\mathbf{P}$ contains preference rank-orders. Gifi (1990) proposes to minimize the closely related problem $\sum_{i=1}^{N} ||\mathbf{X} - \widehat{\mathbf{p}}_i\mathbf{y}_i'||^2$, where $\mathbf{y}_i$ is row i of $\mathbf{Y}$ and $\widehat{\mathbf{p}}_i$ has the same rank-order as $\mathbf{p}_i$ but is optimally transformed. This resembles the strategy for conditional unfolding for the ideal-point model, except that in this case the data are treated as column conditional. To avoid the degenerate solution of $\mathbf{Y} = \mathbf{0}$, $\mathbf{X} = \mathbf{0}$, and $\widehat{\mathbf{p}}_i = \mathbf{0}$, Gifi (1990) imposes the normalization constraint $\widehat{\mathbf{p}}_i'\widehat{\mathbf{p}}_i = n$ and $\mathbf{X}'\mathbf{X} = n\mathbf{I}$. This model can be computed by the program CATPCA (categorical principal components analysis) formerly known under the name PRINCALS (nonlinear principal components analysis), both available in the SPSS package. Note that CATPCA has to be applied to the objects $\times$ individuals matrix, because the ordinal transformations are computed columnwise. More details about this and related approaches can be found in Gifi (1990).

Van Deun et al. (2005) discuss the VIPSCAL model for unfolding. This model allows some subjects to be presented by an ideal point and others by the vector model. The model also allows some length and orthogonality constraints on $\mathbf{X}$ and $\mathbf{Y}$. Special cases within VIPSCAL are the ordinary ideal point model and an (ordinal) vector model.

An Illustrative Application of the Vector Model

Consider the breakfast data in Table 14.1 again. Figure 16.3 shows the result of the vector model for unfolding obtained by CATPCA, using the preference rank-orders only. The preference vectors for every individual are scaled to have equal length, because it is the direction that matters, not the actual length. Note that high values in Table 14.1 indicate least preferred breakfast items; hence the correlations of $\widehat{\mathbf{p}}_i$ with $\mathbf{X}$ (called component loadings in CATPCA) have to be multiplied by minus one to obtain the preference vectors in Figure 16.3. The CATPCA solution indicates that there are three groups of individuals. The first group of 15 respondents is represented by the preference vectors directed away from A. This group has a strong dislike for A (toast pop-up) and does not care much about the other breakfast items either. The other groups are orthogonally related to the first group, indicating that they are indifferent to breakfast A, because A projects onto the origin. The second group is directed to the lower left-hand corner. This group prefers the breakfast items K, D, L, M, and N, and dislikes breakfast items with toast, that is, B, G, and J. The third group has the opposite preference of the second group. The interpretation of this solution is not very different from the ideal-point solution in Figure 14.2.

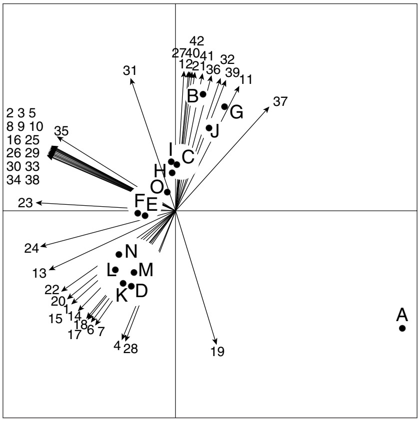

FIGURE 16.3. The vector model of unfolding of the breakfast items in Table 14.1 computed by CATPCA.

16.3 Weighted Unfolding

We now consider a generalization of external unfolding, that is, weighted unfolding (Carroll, 1980). Assume that the coordinate axes could be chosen to correspond to the dimensions that determined person i's preference judgment. It is then possible to conjecture that person i weights these dimensions in some particular way depending on how important he or she feels each dimension to be. Consider, for example, an investment problem and assume that various portfolios are distinguished with respect to risk and expected profit. All individuals agree, say, that portfolio x is riskier than y, and that y has a higher expected yield than z; that is, all individuals perceive the portfolios in the same way. But person i may be more cautious than j, so in making a preference judgment the subject weights the risk dimension more heavily than j. In other words, in making preference judgments on the basis of a common similarity space, person i stretches this space along the risk dimension, but j compresses it, and this will, of course, affect the the distances differentially. We can express such weightings of dimensions as follows.

$$\begin{aligned} d_{ij}(\mathbf{X};\mathbf{Y};\mathbf{W}) &= \left[\sum_{a=1}^{m}(w_{ia}y_{ia} - w_{ia}x_{ja})^2\right]^{1/2} \\ &= \left[\sum_{a=1}^{m} w_{ia}^2(y_{ia} - x_{ja})^2\right]^{1/2}, \end{aligned} \tag{16.3}$$

where x_{ja} is the coordinate of object j on dimension a, y_{ia} is the coordinate of the ideal points for individual i on dimension a, and w_{ia} is the weight that this individual assigns to dimension a.

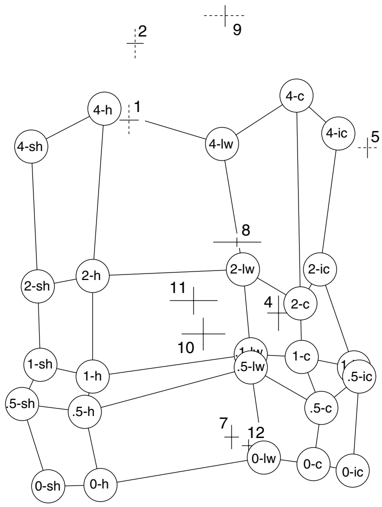

FIGURE 16.4. MDS configuration for tea proximities (circles); numbers indicate teaspoons of sugar, letters temperature of tea (sh="steaming hot", h="hot", lw="lukewarm", c="cold", ic="ice cold"). Crosses show ideal points for ten subjects; length of bars proportional to dimensional weights; dashed/solid bars indicate negative/positive weights, respectively (after Carroll, 1972).

Private Preference Spaces and Common Similarity Space

This seemingly minor modification of the distance formula has important consequences. The most obvious one is that the weighted model generally does not permit the construction of a joint space of objects and individuals in which the differences among the various individuals are represented by the different locations of the respective ideal points. Rather, each individual has his or her own *private preference space*, independent of the preference spaces for other individuals, even though they are all related to a *common similarity space* by *dimensional stretchings*. Further implications of the weighted unfolding model can be seen from the following example.

In an experiment by Wish (see Carroll, 1972), 12 subjects evaluated 25 stimuli with respect to (a) their dissimilarities and (b) their subjective values. The dissimilarity data were collected by rating each of the stimulus pairs on a scale from 0 (= identical) to 9 (= extremely different). The stimuli were verbal descriptions of tea, varying in temperature and sweetness. The proximities are represented by the MDS configuration in Figure 16.4, where the different teas are shown by circles. The configuration **X** reflects the 5×5 design of the stimuli very clearly: the horizontal axis corresponds to the temperature factor and the vertical one to the sweetness scale.

Additionally, the individuals indicated their preferences for each type of tea. These data and the fixed coordinates **X** of the stimuli are used to find the dimension weights and the ideal points for each individual i. To do this, Carroll (1972) minimized the loss function

$$L(\mathbf{Y};\mathbf{W}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(d_{ij}^2(\mathbf{Y};\mathbf{X};\mathbf{W}) - \delta_{ij}^2\right)^2 \tag{16.4}$$

over ideal points $\mathbf{Y}$ and dimension weights $\mathbf{W}$. $L(\mathbf{Y};\mathbf{W})$ differs from a Stress-based criterion in that it uses squared distances d_{ij}^2 for computational convenience instead of the distance d_{ij} (just as in S-Stress; see Section 11.2). $L(\mathbf{Y};\mathbf{W})$ is minimized in an alternating least-squares fashion, where the update of $\mathbf{Y}$ with $\mathbf{W}$ fixed is alternated by the update of $\mathbf{W}$ for fixed $\mathbf{Y}$, until convergence is reached.

Figure 16.4 also represents the resulting 12 private preference spaces through weight crosses on the ideal points. The scatter of the ideal points shows that the individuals differ considerably with respect to the preferred sweetness of the tea. There is much less variation on the temperature dimension, but, strangely, most individuals seem to prefer lukewarm tea, because the ideal points are concentrated mostly in the lukewarm range of the temperature dimension. On the other hand, it is not very surprising that no individual preferred steaming hot tea, and the inclusion of these choice objects might have obscured the situation in Figure 16.4, according to Carroll (1972). He therefore eliminated the steaming hot stimuli from the analysis. This led to an unfolding solution very similar to Figure 16.4 but, of course, without the "sh" points. Its ideal points were still in the lukewarm range, but now the (squared) dimension weights on the temperature dimension were negative for all individuals.

Negative Dimension Weights and Anti-Ideal Points

How are we to interpret negative dimension weights w_{ia}^2? Assume that a given object is considered "ideal" on all dimensions except for dimension a. Then, all dimensional differences are zero in (16.3), except for the one on a. If $w_{ia}^2 < 0$, the term under the square root will be negative. Hence, d_{ij}^2 is negative, and d_{ij} is an imaginary number. But then d_{ij} is not a distance, because distances are nonnegative real numbers, by definition. Thus, without any restrictions on the dimension weights, the weighted unfolding model is not a distance model.

Is such a model needed? Assume that we have a 2D configuration, with person i's ideal point at the origin, and dimension weights $w_{i1}^2 = 1$ and $w_{i2}^2 = -1$. Then, according to (16.3), all points on the bisector between dimensions 1 and 2 have distance zero to the ideal point y_i and, thus, are also ideal points. For *all* points x on, below, and above the bisector, we get $d^2(x, y_i) = 0$, $d^2(x, y_i) > 0$, and $d^2(x, y_i) < 0$, respectively. The plane thus becomes discontinuous and thereby incompatible with the ideal-point model that underlies unfolding. In such a situation, it remains unclear what purpose further generalizations of this model might serve (Srinivasan & Shocker, 1973; Roskam, 1979b; Carroll, 1980).

Should one preserve the idea of negative dimension weights? Carroll (1972) writes: "This possibility of negative weights might be a serious problem except that a reasonable interpretation attaches to negative w's ... This interpretation is simply that if w_{it} [corresponding to our w_{ia}^2] is negative, then, with respect to dimension t, the ideal point for individual i indicates the *least preferred* rather than the most preferred value, and the farther a stimulus is *along that dimension* from the ideal point, the more highly preferred the stimulus" (p. 133). Coombs and Avrunin (1977) argue, however, that *anti-ideal points* are artifacts caused by confounding two qualitatively different sets of stimuli. For tea, they argue that one should expect single-peaked preference functions over the temperature dimension for each iced tea and for hot tea, respectively. For iced tea, each individual has some preferred coldness, and the individual's preference drops when the tea becomes warmer or colder. The same is true for hot tea, except that the ideal temperature for hot tea lies somewhere in the "hot" region of the temperature scale. Thus, iced tea and hot tea both yield single-peaked preference functions over the temperature dimension. Superimposing these functions—and thus generating a meaningless value distribution for "tea"—leads to a two-peaked function with a minimum at lukewarm.

If one restricts the dimension weights to be nonnegative, then there are two models. If zero weights are admitted, d_{ij} in (16.3) is not a distance, because it can be zero for different points. This characteristic means that one cannot interpret the formula as a psychological model saying that person i generates his or her preferences by computing weighted distances in a common similarity space. Rather, the model implies a two-step process, where the individual first weights the dimensions of the similarity space and then computes distances from the ideal point in this ("private") transformed space.

In summary, sensible dimensional weighting allows for better accounting of individual differences, but it also means giving up the joint-space property of simple unfolding. In most applications so far, it turned out that the weighted unfolding model fitted the data only marginally better, and so "relatively little appears to be gained by going beyond the simple (equal-axis weighting) ideal-point model" (Green & Rao, 1972, p.113).

16.4 Value Scales and Distances in Unfolding

We now return to internal unfolding and the simple unfolding model. So far, not much attention has been paid to the exact relationship of the distances between ideal points and object points and the subjective value of the represented objects. We simply claimed that preference strength is linearly or monotonically related to the unfolding distances. The particular shape of the preference function was not derived from further theory. We now consider a model that does just that.

Relating Unfolding Distances to Preference Strength Data by the BTL Model

There are many proposals for modeling preference behavior and subjective value (see, e.g., Luce & Suppes, 1963). One prominent proposal is the Bradley–Terry–Luce (BTL) model (Luce, 1959). This model predicts that person i chooses an object o_j over an object o_k with a probability $p_{jk|i}$ that depends only on the pair (o_j, o_k), not on what other choice objects there are. Restricting the set of choice objects to those that are neither always chosen nor never chosen, a subjective-value scale v can be constructed for i by first selecting some object o_a as an "anchor" of the scale, and then setting

$$v_i(o_j) = \frac{p_{ja|i}}{p_{aj|i}}. \tag{16.5}$$

Conversely, pairwise choice probabilities can be derived from the ratio scale values by using

$$p_{jk|i} = \frac{v_i(o_j)}{v_i(o_j) + v_i(o_k)}. \tag{16.6}$$

Given a set of preference frequencies, it is possible to first find v-values for the choice objects and then map these values into the distances of an unfolding representation. This permits one to test a choice theory (here, the BTL theory) as a first step of data analysis. If the test rejects the choice theory, then it makes little sense to go on to unfolding, because the choice process has not been understood adequately and must be modeled differently. If, on the other hand, the test comes out positive, the distance representation has a better justification.

Luce (1961) and Krantz (1967) discuss two functions that connect the scale v with corresponding distances in the unfolding space. One would want such a function to be monotonically decreasing so that greater v-scale values are related to smaller distances. One reasonable function in this family is

$$d(x_j, y_i) = -\ln[v_i(o_j)], \tag{16.7}$$

or, expressed differently,

$$v_i(o_j) = \exp[-d(x_j, y_i)], \tag{16.8}$$

where $d(x_j, y_i)$ denotes the distance between the points x_j and y_i representing object o_j and individual i, respectively, in the unfolding space. Thus, $v_i(o_j) = \max = 1$ if $d(x_j, y_i) = 0$ (i.e., at the ideal point) and $0 < v_i(o_j) < 1$

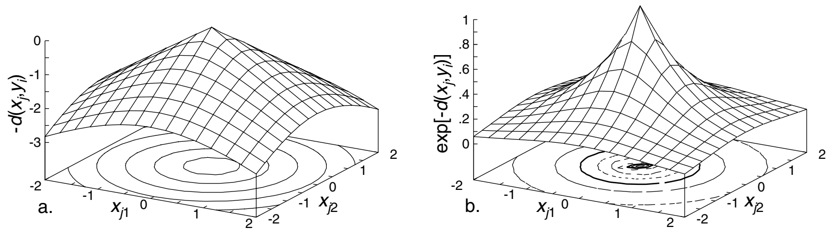

FIGURE 16.5. (a.) Distance (vertical) from ideal point y_i with coordinates (0,0) in 2D to stimulus x_j, and (b) the corresponding scale value $v_i(o_j) = \exp[-d(x_j, y_i)]$ according to (16.8).

for all other objects. Figure 16.5b shows the the preference function for individual i with ideal point (0, 0) in 2D. The circles in the horizontal plane indicate the positions of the object points with equal preference. The corresponding preference strength is shown on the vertical axis. The model defines an inverted bowl over the plane, and this bowl never touches the plane, even though it comes very close to it when we move far away from the ideal point.

A similar function is discussed by Schönemann and Wang (1972) and Wang, Schönemann, and Rusk (1975):

$$v_i(o_j) = \exp[-c \cdot d^2(x_j, y_i)], \tag{16.9}$$

where $c > 0$ is some arbitrary multiplier. Setting $c = 1$, the only difference[3] between (16.9) and (16.8) is that the distances are squared in the former case. For squared distances, the value surface over the object space is normal for each ideal point y_j. Thus, in a 2D case like the one in Figure 16.5b, the inverted bowl has the familiar bell shape. Equation (16.9) is then connected to individual i's pairwise preference probabilities $p_{jk|i}$ by using the BTL choice model. Inserting the $v_i(o_j)$ values into (16.9) yields

$$p_{jk|i} = \frac{1}{1 + \exp[d^2(x_j, y_i) - d^2(x_k, y_i)]^2}. \tag{16.10}$$

Thus, preference probabilities and (squared) distances of the unfolding space are related, according to this model, by a logistic function[4] operating on differences of (squared) distances.

[3]That difference, however, is critical, because it renders the model mathematically tractable so that the exact case can be solved algebraically, without iteration. The algebraic solution given in Schönemann (1970) generalizes the Young–Householder theorem to the asymmetric case.

[4]The exact form of the probability distribution is not of critical importance for fitting the model to the data. This follows from many detailed investigations on generalized Fechner scales (see, e.g., Baird & Noma, 1978), which include the logistic function as just one special case. An alternative is the normal curve, but almost any other approximately symmetrical function would do as well.

TABLE 16.2. Politicians and interviewee groups of Wang et al. (1975) study.

1. Wallace (Wal)	5. Humphrey (Hum)	9. Nixon (Nix)
2. McCarthy (McC)	6. Reagan (Rea)	10. Rockefeller (Roc)
3. Johnson (Joh)	7. Romney (Rom)	11. R. Kennedy (Ken)
4. Muskie (Mus)	8. Agnew (Agn)	12. LeMay (LeM)

	Interviewee Group	Code	N_i
1.	Black, South	BS	88
2.	Black, Non-South	BN	77
3.	White, strong Democrat, South, high ed.	SDSH	17
4.	White, strong Democrat, South, low education	SDSL	43
5.	White, weak Democrat, South, high education	WDSH	27
6.	White, weak Democrat, South, low education	WDSL	79
7.	White, strong Democrat, Non-South, high ed.	SDNH	21
8.	White, strong Democrat, Non-South, low ed.	SDNL	85
9.	White, weak Democrat, Non-South, high ed.	WDNH	65
10.	White, weak Democrat, Non-South, low ed.	WDNL	180
11.	White, Independent, South, high education	ISH	8
12.	White, Independent, South, low education	ISL	27
13.	White, Independent, Non-South, high ed.	INH	25
14.	White, Independent, Non-South, low ed.	INL	46
15.	White, strong Republican, South, low ed.	SRSL	13
16.	White, strong Republican, Non-South, high ed.	SRNH	40
17.	White, strong Republican, Non-South, low ed.	SRNL	60
18.	White, weak Republican, South, high ed.	WRSH	34
19.	White, weak Republican, South, low ed.	WRSL	36
20.	White, weak Republican, Non-South, high ed.	WRNH	90
21.	White, weak Republican, Non-South, low ed.	WRNL	117

An Application of Schönemann and Wang's BTL Model

Consider an application. Wang et al. (1975) analyzed data collected in 1968 on 1178 persons who were asked to evaluate 12 candidates for the presidency on a rating scale from 0 (= very cold or unfavorable feeling for the candidate) to 100 (= very warm or favorable feeling toward the candidate) (Rabinowitz, 1975). The respondents were classified into 21 groups according to their race, party preference, geographical region, and education. The 21 groups and the 12 candidates are listed in Table 16.2. Twenty-one 12×12 preference matrices were derived from the rating values of the respondents in each group. The $p_{jk|i}$ values (where i indicates the group $i = 1, \ldots, 21$) were computed as the relative frequencies with which candidate o_j's rating score was higher than the score for candidate o_k.

The least-squares BTL scale values for the 12 candidates and the 21 groups are shown in Table 16.3. It turned out that these scale values accounted for the probabilities sufficiently well; that is, it is possible to approximately reconstruct the $\binom{12}{2}$ probability data from the 12 scale values for each group. By taking the logarithm of both sides of (16.9), the v-values can be transformed into squared distances, which in turn are the dissimilarities for our unfolding analysis. Wang et al. (1975) then employed an iterative optimization method for finding an unfolding configuration. (Of course, the internal unfolding solution could also be computed by the majorization algorithm in Section 14.2.) The final fit to the $i = 1, \ldots, 21$ em-

TABLE 16.3. BTL scale values for interviewee groups and politicians from Table 16.2.

	Wal	Hum	Nix	McC	Rea	Roc	Joh	Rom	Ken	Mus	Agn	LeM
BS	.11	9.00	.95	.75	.28	.66	9.10	.51	21.70	1.52	.35	.14
BN	.03	12.09	.95	1.27	.29	1.78	7.54	.58	16.02	2.15	.26	.11
SDSH	.49	3.82	1.00	1.17	.42	.94	1.48	.45	1.89	3.43	.62	.43
SDSL	.86	2.64	1.09	.58	.46	.70	2.54	.53	1.89	1.70	.74	.67
WDSH	.72	1.08	2.82	1.01	.84	1.21	1.27	.56	1.37	1.52	.69	.44
WDSL	1.24	1.20	2.30	.76	.68	.66	1.12	.64	1.45	.93	1.03	.86
SDNH	.09	4.72	1.11	1.67	.40	1.07	2.64	1.23	5.92	4.44	.38	.09
SDNL	.26	3.80	.95	.86	.43	.81	3.12	.64	5.99	2.38	.49	.25
WDNH	.12	2.99	1.46	1.68	.42	1.61	1.54	.92	5.13	2.46	.49	.19
WDNL	.37	1.99	1.57	.98	.59	.82	1.58	.68	3.69	1.71	.70	.40
ISH	.43	1.24	4.07	.76	.89	.97	.88	.60	2.88	1.10	1.34	.31
ISL	6.68	.90	3.40	.87	.83	.86	.95	.51	2.53	.87	1.03	.71
INH	.43	1.77	2.54	1.49	.66	.84	1.01	.77	1.69	1.80	.88	.30
INL	6.37	1.48	2.30	1.13	.74	.95	1.12	.71	2.66	1.82	.83	.31
SRSL	.11	.66	20.37	.55	1.43	.82	.86	.78	1.93	.77	3.20	.34
SRNH	.16	.55	14.29	1.05	1.98	1.44	.54	1.29	1.26	.98	1.19	.26
SRNL	.28	.62	8.49	1.02	1.39	1.02	.61	.95	1.28	.87	1.78	.41
WRSH	.76	.45	7.53	.64	1.78	.99	.82	.65	.86	.82	1.15	.80
WRSL	.85	.56	5.23	1.07	1.03	1.10	.78	.62	1.55	.55	1.23	.66
WRNH	.28	.89	4.78	1.56	1.12	1.46	.68	.75	1.38	1.34	1.01	.34
WRNL	.33	.86	5.84	1.06	1.08	1.06	.76	.73	1.75	.99	1.17	.43

pirical preference probabilities can be checked by substituting the $d^2(x_j, y_i)$ terms in (16.10) with the reconstructed distances in the unfolding solution. Wang et al. (1975) concluded from statistical tests that a 3D representation was sufficiently precise.

The 3D unfolding representation, however, possesses a peculiar property: the ideal points are not distributed throughout the whole space, but lie almost completely in a plane. This implies that the solution has a considerable indeterminacy with respect to the point locations.[5] Figure 16.6 illustrates the problem with a 2D example. All ideal points $y_1, \ldots, y_4$ lie on a straight line, whereas the object points $x_1, \ldots, x_5$ scatter throughout the space. In internal unfolding, the only information available for determining the location of the points is the closeness of object and ideal points. But then each x_j can be reflected on the line running through y_is, because this does not change any between-sets distance. Thus, for example, instead of the solid point x_2 in Figure 16.6, we could also choose its counterpoint shown as an open circle. Such choices have a tremendous effect on the appearance of the unfolding solution and, by way of that, on its interpretation.

How can one diagnose this subspace condition in practice? One can do a principal axes rotation of the **X** configuration and of the **Y** configuration, respectively, and then check, on the basis of the eigenvalues, whether either one can be said to essentially lie in a subspace of the joint space. Table 16.4

[5]This indeterminacy is not restricted to the Schönemann and Wang model, but it is a property of all Euclidean ideal-point unfolding models.

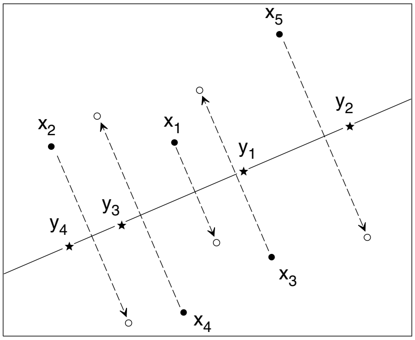

FIGURE 16.6. An indeterminacy in an unfolding space; the x_j points can be reflected on the line of the y_is without affecting the distance $d(x_j, y_i)$.

TABLE 16.4. Coordinates for candidates and subgroup ideal points in 3D: unrotated (left-hand side, **X** and **Y**) and after subspace rotation (right-hand side, $\mathbf{X}^*$ and $\mathbf{Y}^*$).

X	1	2	3	**Y**	1	2	3	$\mathbf{X}^*$	1	2	3	$\mathbf{Y}^*$	1	2	3
Wal	-1.17	-1.40	-.31	NS	.66	.21	-.39	Wal	.99	-1.38	-.97	NS	-.10	.63	-.14
Hum	1.41	-.31	.24	NN	.74	.51	-.42	Hum	.78	1.24	.14	NN	-.40	.75	-.00
Nix	-1.21	.15	-.09	SDSH	.21	-.53	.29	Nix	-.20	-1.25	.05	SDSH	.80	.03	.04
McC	.07	.53	1.42	SDSL	.14	-.59	.20	McC	.49	-.09	1.56	SDSL	.79	-.04	-.08
Rea	-.93	.56	1.19	WDSH	-.06	-.22	.10	Rea	.18	-1.05	1.35	WDSH	.40	-.18	.03
Roc	-.03	.63	1.39	WDSL	-.11	-.43	.12	Roc	.37	-.18	1.58	WDSL	.57	-.26	-.07
Joh	1.02	-.89	-.80	SDNH	.48	.22	-.08	Joh	.66	.89	-1.06	SDNH	.02	.42	.13
Rom	-.12	.71	1.47	SDNL	.37	-.16	-.02	Rom	.32	-.26	1.69	SDNL	.35	.26	-.02
Ken	1.16	-.46	-.54	WDNH	.32	.20	-.10	Ken	.46	1.06	-.61	WDNH	.00	.26	.09
Mus	.53	.29	1.28	WDNL	.15	-.12	-.03	Mus	.69	.34	1.31	WDNL	.28	.05	-.01
Agn	-.96	-.89	-1.05	ISH	-.05	.22	-.26	Agn	.21	-1.04	-1.31	ISH	-.16	-.08	-.04
LeM	-.96	-1.34	-.91	ISL	-.11	-.06	-.12	LeM	.66	-1.11	-1.44	ISL	.14	-.19	-.07
				INH	.03	-.07	.03					INH	.25	-.07	.05
				INL	.06	.01	-.05					INL	.15	-.02	.03
				SRSL	-.25	1.14	-.92					SRSL	-1.31	-.09	-.10
				SRNH	-.27	.85	-.50					SRNH	-.85	-.19	.09
				SRNL	-.26	.53	-.39					SRNL	-.52	-.23	.01
				WRSH	-.34	.09	-.15					WRSH	-.04	-.40	-.02
				WRSL	-.25	.03	-.14					WRSL	.02	-.32	-.04
				WRNH	-.13	.28	-.15					WRNH	-.16	-.16	.09
				WRNL	-.15	.32	-.27					WRNL	-.26	-.17	.01

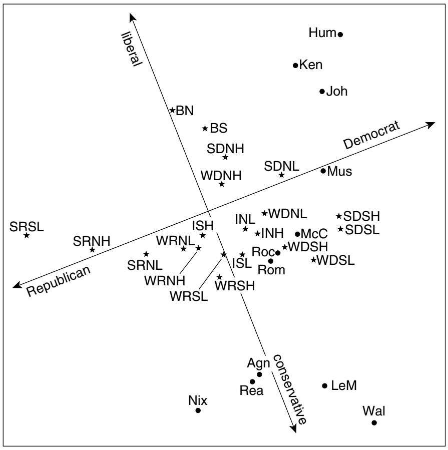

FIGURE 16.7. Unfolding representation of BTL values in Table 16.3; for labels, see Table 16.2 (after Wang et al., 1975).

illustrates this approach for the Wang et al. case. The panels on the left-hand side show the coordinates for the candidates and the ideal points in some 3D joint space. The panels on the right-hand side exhibit the coordinates of both configurations in a rotated 3D joint space whose dimensions correspond to the principal axes of the ideal-point configuration. The table (rightmost column) shows that the ideal points lie essentially in a plane of the 3D joint space, because their coordinates on the third principal axis are all very similar (here: close to zero). If one projects the candidates into this plane, one obtains an unfolding representation that is free from reflection indeterminacies.

Figure 16.7 represents the person groups as stars and the candidates as points. The dimensions correspond to the interpretation given by Wang et al. (1975) on the basis of considering the projections of the candidates onto various straight lines. But one could also proceed by studying the ideal-point labels. For example, a Republican vs. Democrat dimension is suggested by studying the party affiliations of the various groups of white voters. If we draw lines around the groups with party preference SR, WR, I, WD, and SD, respectively, regions of ideal points result that can be partitioned almost perfectly by parallel straight lines. These lines are, however, not quite orthogonal to the direction of the Republican–Democrat dimension chosen by Wang et al. Rather, they partition the axis through the points Nixon and Humphrey. The two other group facets, education and region, do not allow simple partitionings. Wang et al.'s liberal–conservative dimension essentially distinguishes blacks from whites.

Interpreting dimensions is not affected by the reflection indeterminacy discussed above. In contrast, the usual ideal-point interpretation is only partially possible in Figure 16.7. A naive approach can lead to gross mistakes. We know from Table 16.4 that a number of candidate points such as Rockefeller, for example, are positioned far above or below the subspace plane shown in Figure 16.7. But Figure 16.7 shows "Roc" close to most ideal points, so that one might expect, incorrectly, that Rockefeller is among the top choices for most groups. The usual ideal-point unfolding interpretation leads to correct preference predictions only for those candidates close to the subspace plane, such as Nixon and Humphrey.

How should one interpret such extra dimensions? There is no simple answer, and additional information beyond the data on which the unfolding is based is required in any case. In the given example, one might speculate that the extra dimension for the candidates reflects additional features of the candidates, unrelated to the preferential choice criteria that distinguish the different groups, such as, for example, the extent to which the candidates are known or unknown. In any case, such interpretations remain complicated because each point can be reflected on this dimension.

Although the meaning of the joint space and its ideal-point subspace remains somewhat unclear in the given example, it is easy to derive some testable implications. The BTL model states a function between v-values of objects and the probability for choosing one object o_j out of any set of choice objects. This function is simply the v-value of object o_i divided by the sum of the v-values of all choice objects. The v-values, in turn, can be estimated from the unfolding distances using (16.9). For the three candidates Nixon, Humphrey, and Wallace (i.e., those that actually remained as candidates in the general presidential election) we can thus estimate, for each group, the probability for choosing each candidate out of the three remaining ones. The prediction for a candidate's chances in the general election is then the (weighted) average of all 21 group-specific probabilities. The predicted preference probabilities of voting for Wallace, Humphrey, or Nixon, computed in this way, are 0.0797, 0.3891, and 0.5311, respectively. These values are quite close to the relative frequencies of direct votes given in the interviews, which are 0.1091, 0.4122, and 0.4788, respectively.

16.5 Exercises

Exercise 16.1 Consider the vector model for unfolding.

(a) First, set up a configuration $\mathbf{X}$ such as the one shown in the table below. Then, define preference vectors for a number of persons, $\mathbf{p}_i$ $(i = 1, \ldots)$, as lines that run through the origin $E = (0, 0)$ and through one other point (x_{1i}, x_{2i}) of $\mathbf{X}$. Finally, construct the preference scale for each person i by projecting the points of $\mathbf{X}$ onto the ideal vectors.

Object	Dim. 1	Dim. 2
A	-1	1
B	0	1
C	1	1
D	-1	0
E	0	0
F	1	0
G	-1	-1
H	0	-1
I	1	-1

(b) Discuss, in terms of psychology, the meaning of the coordinates y_{i1} and y_{i2} of each person i. What do these "weights" express? (Hint: How much do the dimensions of $\mathbf{X}$ contribute to an ideal line's direction?)

(c) How should y_{i1} and y_{i2} be restricted in model (16.2)? (Hint: Note the constraint on $\text{diag}(\mathbf{YY}')$. How can you interpret the thus-constrained coordinates?

(d) Unfold the preference data thus constructed and compare the solution to the $\mathbf{X}$ and the $\mathbf{Y}$ from which you started.

(e) Add random error to $\mathbf{X}$ and $\mathbf{Y}$ and repeat the above investigations for different levels of error. Discuss the robustness of the scaling procedure.

(f) Construct a preference vector that does *not* fit into the space of the objects, $\mathbf{X}$. What could you do to represent it in the preference vector model anyway? (Hint: Consider augmenting the dimensionality of the unfolding space.)

Exercise 16.2 Consider the country-by-attributes data in Exercise 15.1.

(a) Discuss the ideal-point unfolding model for these data. How does it differ from scaling the proximities for the countries (as in Section 1.3) and then fitting external property scales (as in Section 4.3)?

(b) Discuss the difference between an ideal-point model and a vector model in unfolding preferential data and what this difference means in the context of the attribute-by-country data.

(c) Scale the country-by-attributes data into a vector unfolding model, with countries as points and attributes as vectors. Then, scale the same data into an ideal-point model. Compare the solutions in terms of what they suggest about how the student-subjects perceived these countries.

(d) Would it make sense to also scale the countries into vectors, and the attributes into points? How would you interpret such a solution?

Exercise 16.3 The following data set is a data set reported by SAS (1999). It contains the ratings by 25 judges of their preference for each of 17 automobiles. The ratings are made on a 0 to 9 scale, with 0 meaning very weak preference and 9 meaning very strong preference for the automobile.

	Manufacturer	Type	Rating per Judge
1	Cadillac	Eldorado	8 0 0 7 9 9 0 4 9 1 2 4 0 5 0 8 9 7 1 0 9 3 8 0 9
2	Chevrolet	Chevette	0 0 5 1 2 0 0 4 2 3 4 5 1 0 4 3 0 0 3 5 1 5 6 9 8
3	Chevrolet	Citation	4 0 5 3 3 0 5 8 1 4 1 6 1 6 4 3 5 4 4 7 4 7 7 9 5
4	Chevrolet	Malibu	6 0 2 7 4 0 0 7 2 3 1 2 1 3 4 5 5 4 5 6 6 8 6 5 8
5	Ford	Fairmont	2 0 2 4 0 0 6 7 1 5 0 2 1 4 4 3 5 3 0 6 4 8 6 5 5
6	Ford	Mustang	5 0 0 7 1 9 7 7 0 5 0 2 1 1 0 1 8 5 0 6 5 7 5 5 5
7	Ford	Pinto	0 0 2 1 0 0 0 3 0 3 0 3 0 2 0 1 5 0 0 5 1 4 0 7 8
8	Honda	Accord	5 9 5 6 8 9 7 6 0 9 6 9 9 9 5 2 9 9 8 9 7 5 0 7 8
9	Honda	Civic	4 8 3 6 7 0 9 5 0 7 4 8 8 8 5 2 5 6 7 7 6 5 0 7 5
10	Lincoln	Continental	7 0 0 8 9 9 0 5 9 2 2 3 0 4 0 9 9 6 2 0 9 1 9 0 9
11	Plymouth	Gran Fury	7 0 0 6 0 0 0 4 3 4 1 0 1 1 0 7 3 3 3 4 5 8 7 0 8
12	Plymouth	Horizon	3 0 0 5 0 0 5 6 3 5 4 6 1 3 0 2 4 4 4 6 7 5 6 5 5
13	Plymouth	Volare	4 0 0 5 0 0 3 6 1 4 0 2 1 6 0 2 7 5 4 4 7 6 5 5 5
14	Pontiac	Firebird	0 1 0 7 8 9 5 6 1 3 2 0 1 2 0 6 9 5 8 2 6 5 9 0 7
15	Volkswagen	Dasher	4 8 5 8 6 9 6 5 0 8 8 7 7 7 9 5 3 7 7 8 9 5 0 0 0
16	Volkswagen	Rabbit	4 8 5 8 5 0 9 7 0 9 6 9 5 7 9 5 4 8 7 8 8 5 0 0 0
17	Volvo	DL	9 9 8 9 9 9 8 9 0 9 9 9 9 9 8 7 9 8 9 9 1 9 0 0 0

(a) Unfold these preference data into the vector model, with cars as points and vectors as persons. Discuss the solution in terms of what it says about the different automobiles, and what it suggests about groups of potential buyers of automobiles and their preferences.

(b) It was previously observed from unfolding these data that the solution "suggests that there is a market for luxury Japanese and European cars" (http://rocs.acomp.usf.edu/sas/sashtml/stat/chap53/sect25.htm). How did the market researchers arrive at this insight? On what assumptions does this interpretation hinge? Would you be willing to bet your money on this interpretation?

Exercise 16.4 Use the data in Table 16.3 on p. 349to construct a vector-model unfolding representation. Compare your solution to the configuration in Figure 16.7. Discuss where the models suggest similar substantive conclusions (despite possibly different "looks" of the plots), and where they differ.

Exercise 16.5 The table below shows the (contrived) preferences of six different persons for the composition of an ideal family in terms of how many children a person wants, and whether these children should be girls or boys. For example, person 1 wants no children at all, and his or her second choice is one boy. Person 2, on the other hand, ideally wants 2 girls and 2 boys.

Number of		Person					
Girls	Boys	1	2	3	4	5	6
0	0	1	9	5	6	6	7
0	1	2	8	3	7	2	3
0	2	5	5	1	9	7	8
1	0	3	7	8	2	3	4
1	1	4	4	4	4	1	1
1	2	8	2	2	8	4	5
2	0	6	6	9	1	8	9
2	1	7	3	7	3	5	2
2	2	9	1	6	5	9	6

(a) Use ordinal unfolding to study the structure of these preference data. Some programs and some model specifications are likely to yield degenerate solutions. Is your solution degenerate? If so, can you prevent this degeneracy?

(b) The space of choice objects and its dimensions can be thought of as a "boys by girls" space. Does your unfolding yield this space?

(c) Experiment with constraints on the unfolding model so that the boys-by-girls configuration in its solution space approximates a rectangular grid pattern.

(d) Although such family composition preference data have been analyzed before within an unfolding framework, the unfolding model is not really adequate for them. Why? (Hint: Can you have a preference for 1.3 boys and 2.8 girls, for example? Take a close look at the ideal-point isopreference-contours model.)