The Structural Representation of Proximity Matrices With MATLAB
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Preface

As the title of this monograph implies, our main goal is to provide and illustrate the use of functions (by way of m-files) within a MATLAB\textsuperscript{1} computational environment to effect a variety of structural representations for proximity information assumed available on a set of objects. The structural representations that will be of interest have been discussed and developed primarily in the applied (behavioral science) statistical literature (e.g., in psychometrics and classification), although interest in these topics has now extended much more widely (for example, to bioinformatics and chemometrics). We subdivide the monograph into three main sections depending on the general class of representations being discussed. Part I will develop linear and circular uni- and multi-dimensional scaling using the city-block metric as the major representational device; Part II is concerned with characterizations based on various graph-theoretic tree structures, and specifically with those usually referred to as ultrametrics and additive trees; Part III uses representations defined solely by order properties, and particularly to what are called (strongly) anti-Robinson forms. Irrespective of the part of the monograph being discussed, there generally will be two kinds of proximity information analyzed: one-mode and two-mode. One-mode proximity data are defined between the $n$ objects from a single set, and usually given in the form of a square ($n \times n$) symmetric matrix with a zero main diagonal; two-mode proximity data are defined between the objects from two distinct sets containing, say, $n_a$ and $n_b$ objects, respectively, and given in the form of a rectangular ($n_a \times n_b$) matrix. Also, there will generally be the flexibility to allow the fitting (additively) of multiple structures to either the given one- or two-mode proximity information.

It is not the intent of the monograph to present formal demonstrations of the various assertions we might make along the way, such as for the convergence of a particular algorithm or approach. All of this is generally available in the literature (and much of it by the authors of the current monograph), and the references to this source material is given when appropriate. The primary interest here is to present and demonstrate how to actually find and fit these structures computationally with the help of some sixty-five functions (though m-files) we provide that are usable within a MATLAB computational environment. The usage header information for each of these functions is given in Appendix A (listed alphabetically). The m-files themselves can be downloaded individually from

\textsuperscript{1}MATLAB is a registered trademark of The MathWorks, Inc.
Also, there is a “zipped” file called \texttt{srpm\_mfiles.zip} at this site that includes them all, as well as the few small data sets used throughout the monograph to illustrate the results of invoking the various m-files (or we might say, invoking the various “m-functions”); thus, the reader should be able to reproduce all of the examples given in the monograph (assuming, obviously, access to a MATLAB environment).

The computational approach implemented in the provided m-files for obtaining the sundry representations, are by choice, invariably least-squares, and based on what is called the Dykstra-Kaczmarz (DK) method for solving linear inequality constrained least-squares tasks. The latter iterative strategy is reviewed in Chapter 1 (Section 1.4, in particular). All of the representations of concern (over all three monograph Parts) can be characterized by explicit linear inequalities; thus, once the latter constraints are known (by, for example, the identification of certain object permutations through secondary optimization problems such as quadratic assignment), the actual representing structure can be obtained by using the iterative DK strategy. Also, as we will see particularly in Part II dealing with graph-theoretic tree structures (ultrametrics and additive trees), the DK approach can even be adopted heuristically to first identify the inequality constraints that we might wish to impose in the first place. And once identified in this exploratory fashion, a second application of DK could then do a confirmatory fitting of the now fixed inequality constraints.

As noted above, our purpose in writing this monograph is to provide an applied documentation source for a collection of m-files that would be of interest to applied statisticians and data analysts but also accessible to a notationally sophisticated but otherwise substantively focused user. Such a person would typically be most interested in analyzing a specific data set by adopting one (or some) of the structural representations we discuss. The background we have tried to assume is at the same level required to follow the documentation for good, commercially available optimization subroutines, such as the Numerical Algorithms Group (NAG) Fortran subroutine library, or at the level of one of the standard texts in applied multivariate analysis usually used for a graduate second-year methodology course in the behavioral and social sciences. An excellent example of the latter would be the widely used text now in its fifth edition by Johnson and Wichern (2002). Draft versions of the current monograph have been used as supplementary material for a course relying on the latter text as the primary reference.

The research reported in this monograph has been partially supported by the National Science Foundation through Grant No. SES-981407 (to LH), and by the Netherlands Organization for Scientific Research (NWO) through Grant No. 575-67-053 for the ‘PIONEER’ project ‘Subject Oriented Multivariate Analysis’ (to JM).

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Phipps Arabie
Jacqueline Meulman

September, 2004
Part I

(Multi- and Uni-dimensional) City-Block Scaling
Chapter 1
Linear Unidimensional Scaling

The task of linear unidimensional scaling (LUS) can be characterized as a specific data analysis problem: given a set of \( n \) objects, \( S = \{ O_1, ..., O_n \} \), and an \( n \times n \) symmetric proximity matrix, \( P = \{ p_{ij} \} \), arrange the objects along a single dimension such that the induced \( n(n-1)/2 \) interpoint distances between the objects reflect the proximities in \( P \). The term “proximity” refers to any symmetric numerical measure of relationship between each object pair (\( p_{ij} = p_{ji} \) for \( 1 \leq i, j \leq n \)) and for which all self-proximities are considered irrelevant and set equal to zero (\( p_{ii} = 0 \) for \( 1 \leq i \leq n \)). As a technical convenience, proximities are assumed nonnegative and are given a dissimilarity interpretation, so that large proximities refer to dissimilar objects.

As a starting point to be developed exclusively in this first chapter, we consider the most common formalization of measuring how close the interpoint distances are to the given proximities by the sum of squared discrepancies. Specifically, we wish to find the \( n \) coordinates, \( x_1, x_2, ..., x_n \), such that the least-squares (or \( L_2 \)) criterion

\[
\sum_{i<j} (p_{ij} - |x_j - x_i|)^2
\]

is minimized. Although there is some arbitrariness in the selection of this measure of goodness-of-fit for metric scaling, the choice is traditional and has been discussed in some detail in the literature by Guttman (1968), Defays (1978), de Leeuw and Heiser (1977), and Hubert and Arabie (1986), among others. In the various sections that follow, we will develop a particular heuristic strategy for the minimization of (1.1) based on the iterative use of a quadratic assignment improvement technique. Other methods are possible but will not be explicitly discussed here; the reader is referred to Hubert, Arabie, and Meulman (2002) for a comparison among several optimization alternatives for the basic LUS task.

In addition to developing the combinatorial optimization task of actually identifying a best unidimensional scaling, Section 1.3 introduces two additional problems within the LUS context: (a) the confirmatory fitting of a unidimensional scale (through coordinate estimation) based on a fixed (and given) object ordering; (b)
the extension to nonmetric unidimensional scaling incorporating an additional optimal monotonic transformation of the proximities. Both these optimization tasks are formulated through the $L_2$-norm and implemented using applications of what is called the Dykstra-Kaczmarz method of solving linear (in)equality constrained least-squares tasks (Kaczmarz, 1937; Dykstra, 1983) The latter strategy is reviewed briefly in a short addendum (in Section 1.4) to this chapter.

1.1 LUS in the $L_2$-Norm

As a reformulation of the $L_2$ unidimensional scaling task that will prove crucial as a point of departure in our development of a computational routine, the optimization suggested by (1.1) can be subdivided into two separate problems to be solved simultaneously: find a set of $n$ numbers, $x_1 \leq x_2 \leq \cdots \leq x_n$, and a permutation on the first $n$ integers, $\rho(\cdot) \equiv \rho$, for which

$$
\sum_{i<j} (p_{\rho(i)\rho(j)} - (x_j - x_i))^2
$$

is minimized. Thus, a set of locations (coordinates) is defined along a continuum as represented in ascending order by the sequence $x_1, x_2, \ldots, x_n$; the $n$ objects are allocated to these locations by the permutation $\rho$, so that object $O_{\rho(i)}$ is placed at location $i$. Without loss of generality we will impose the one additional constraint that $\sum_i x_i = 0$; i.e., any set of values, $x_1, x_2, \ldots, x_n$, can be replaced by $x_1 - \bar{x}, x_2 - \bar{x}, \ldots, x_n - \bar{x}$, where $\bar{x} = (1/n) \sum_i x_i$, without altering the value of (1.1) or (1.2). Formally, if $\rho^*$ and $x^*_1 \leq x^*_2 \leq \cdots \leq x^*_n$ define a global minimum of (1.2), and $\Omega$ denotes the set of all permutations of the first $n$ integers, then

$$
\sum_{i<j} (p_{\rho^*(i)\rho^*(j)} - (x^*_j - x^*_i))^2 = 
\min \left\{ \sum_{i<j} (p_{\rho(i)\rho(j)} - (x_j - x_i))^2 \mid \rho \in \Omega; \ x_1 \leq \cdots \leq x_n; \ \sum_i x_i = 0 \right\}.
$$

The measure of loss in (1.2) can be reduced algebraically:

$$
\sum_{i<j} p_{ij}^2 + n(\sum_i x_i^2 - 2 \sum_i x_i t_i^{(\rho)}),
$$

subject to the constraints that $x_1 \leq \cdots \leq x_n$ and $\sum_i x_i = 0$, and letting

$$
t_i^{(\rho)} = (u_i^{(\rho)} - v_i^{(\rho)})/n,
$$

where

$$
u_i^{(\rho)} = \sum_{j=1}^{i-1} p_{\rho(i)\rho(j)} \text{ for } i \geq 2;
$$

$$
u_i^{(\rho)} = \sum_{j=i+1}^n p_{\rho(i)\rho(j)} \text{ for } i < n,
$$

and
1.1. LUS in the $L_2$-Norm

and

\[ u_1^{(p)} = v_n^{(p)} = 0. \]

Verbally, $u_1^{(p)}$ is the sum of the entries within row $p(i)$ of \{\$p_{p(i)p(j)}\} from the extreme left up to the main diagonal; $v_1^{(p)}$ is the sum from the main diagonal to the extreme right. Or, we might rewrite (1.3) as

\[ \sum_{i<j} p_{ij}^2 + n \left( \sum_i (x_i - t_i^{(p)})^2 - \sum_i (t_i^{(p)})^2 \right). \tag{1.4} \]

In (1.4), the two terms $\sum_{i<j} (x_i - t_i^{(p)})^2$ and $\sum_i (t_i^{(p)})^2$ control the size of the discrepancy index since $\sum_{i<j} p_{ij}^2$ is constant for any given data matrix. Thus, to minimize the original index in (1.2), we should simultaneously minimize $\sum_i (x_i - t_i^{(p)})^2$ and maximize $\sum_i (t_i^{(p)})^2$. If the equivalent form of (1.3) is considered, our concern would be in minimizing $\sum_i x_i^2$ and maximizing $\sum_i x_i t_i^{(p)}$.

As noted first by Defays (1978), the minimization of (1.4) can be carried out directly by the maximization of the single term, $\sum_i (t_i^{(p)})^2$ (under the mild regularity condition that all off-diagonal proximities in $P$ are positive and not merely nonnegative). Explicitly, if $\rho^*$ is a permutation that maximizes $\sum_i (t_i^{(p)})^2$, then we can let $x_i = t_i^{(p)^*}$, which eliminates the term $\sum_i (x_i - t_i^{(p)^*})^2$ from (1.4). In short, because the order induced by $t_1^{(p^*^*)}, \ldots, t_n^{(p^*^*)}$ is consistent with the constraint $x_1 \leq x_2 \leq \cdots \leq x_n$, the minimization of (1.4) reduces to the maximization of the single term $\sum_i (t_i^{(p)})^2$, with the coordinate estimation completed as an automatic byproduct.

1.1.1 A Data Set for Illustrative Purposes

It is convenient to have a small numerical example available as we discuss optimization strategies in the unidimensional scaling context. Toward this end we list a data file in Table 1.1, called `number.dat`, that contains a dissimilarity matrix taken from Shepard, Kilpatrick, and Cunningham (1975). The stimulus domain is the first ten single-digits \{0,1,2, \ldots, 9\} considered as abstract concepts; the 10 x 10 proximity matrix (with an $i^{th}$ row or column corresponding to the $i$ - 1 digit) was constructed by averaging dissimilarity ratings for distinct pairs of those integers over a number of subjects and conditions. Given the various analyses of this proximity matrix that have appeared in the literature (e.g., see Hubert, Arabie, and Meulman, 2001), the data reflect two types of very regular patterning based on absolute digit magnitude and the structural characteristics of the digits (e.g., the powers or multiples of 2 or of 3, the salience of the two additive/multiplicative identities [0/1], oddness/evenness). These data will be relied on to provide concrete numerical illustrations of the various MATLAB functions we introduce, and will be loaded as a proximity matrix (and importantly, as one that is symmetric and has zero values along the main diagonal) in the MATLAB environment by the command `load number.dat`. As we will see, the dominant single unidimensional scale found for these data is most consistent with digit magnitude.
1.2 L₂ Optimization Methods

This section shows how a well-known combinatorial optimization task, called quadratic assignment, can be used iteratively for LUS in the $L_2$-norm. Based on the reformulation in (1.3), we concentrate on maximizing $\sum_i x_it_i(\rho)$, with iterative re-estimation of the coordinates $x_1, \ldots, x_n$. Various function implementations within MATLAB are given both for the basic quadratic assignment task as well as for how it is used for LUS.

1.2.1 Iterative Quadratic Assignment

Because of the manner in which the discrepancy index for the unidimensional scaling task can be rephrased as in (1.3) and (1.4), the two optimization subproblems to be solved simultaneously of identifying an optimal permutation and a set of coordinates can be separated:

(a) assuming that an ordering of the objects is known (and denoted, say, as $\rho^0$ for the moment), find those values $x_1^0 \leq \cdots \leq x_n^0$ to minimize $\sum (x_i^0 - t_i(\rho^0))^2$. If the permutation $\rho^0$ produces a monotonic form for the matrix $\{p_{\rho^0(i)\rho^0(j)}\}$ in the sense that $t_1^{(\rho^0)} \leq t_2^{(\rho^0)} \leq \cdots \leq t_n^{(\rho^0)}$, the coordinate estimation is immediate by letting $x_i^0 = t_i^{(\rho^0)}$, in which case $\sum_i (x_i^0 - t_i^{(\rho^0)})^2$ is zero.

(b) assuming that the locations $x_1^0 \leq \cdots \leq x_n^0$ are known, find the permutation $\rho^0$ to maximize $\sum_i x_it_i(\rho^0)$. We note from the work of Hubert and Arabie (1986, p. 189) that any such permutation which even only locally maximizes $\sum_i x_it_i(\rho^0)$, in the sense that no adjacently placed pair of objects in $\rho^0$ could be interchanged to increase the index, will produce a monotonic form for the non-negative matrix $\{p_{\rho^0(i)\rho^0(j)}\}$. Also, the task of finding the permutation $\rho^0$ to maximize $\sum_i x_it_i(\rho^0)$ is actually a quadratic assignment (QA) task which has been discussed extensively in the literature of operations research, e.g., see Francis and White (1974), Lawler

Table 1.1. The number.dat data file extracted from Shepard, Kilpatric, and Cunningham (1975)

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</table>
1.2. \( L_2 \) Optimization Methods

(1975), Hubert and Schultz (1976), among others. As usually defined, a QA problem involves two \( n \times n \) matrices \( A = \{a_{ij}\} \) and \( B = \{b_{ij}\} \), and we seek a permutation \( \rho \) to maximize

\[
\Gamma(\rho) = \sum_{i,j} a_{\rho(i)\rho(j)} b_{ij}.
\]

(1.5)

If we define \( b_{ij} = |x_i - x_j| \) and let \( a_{ij} = p_{ij} \), then

\[
\Gamma(\rho) = \sum_{i,j} p_{\rho(i)\rho(j)} |x_i - x_j| = 2n \sum_i x_i t_i^{(\rho)},
\]

and thus, the permutation that maximizes \( \Gamma(\rho) \) also maximizes \( \sum_i x_i t_i^{(\rho)} \).

The QA optimization task as formulated through (1.5) has an enormous literature attached to it, and the reader is referred to Pardalos and Wolkowicz (1994) for an up-to-date and comprehensive review. For current purposes and as provided in three general m-functions of the next section (\texttt{pairwiseqa.m}, \texttt{rotateqa.m}, and \texttt{insertqa.m}), one might consider the optimization of (1.5) through simple object interchange/rearrangement heuristics. Based on given matrices \( A \) and \( B \), and beginning with some permutation (possibly chosen at random), local interchanges/rearrangements of a particular type are implemented until no improvement in the index can be made. By repeatedly initializing such a process randomly, a distribution over a set of local optima can be achieved. At least within the context of some common data analysis applications, such a distribution may be highly relevant diagnostically for explaining whatever structure might be inherent in the matrix \( A \).

In a subsequent subsection below, we introduce the main m-function for unidimensional scaling (\texttt{uniscalqa.m}) based on these earlier QA optimization strategies. In effect, we begin with an equally-spaced set of fixed coordinates with their interpoint distances defining the \( B \) matrix of the general QA index in (1.5) and a random object permutation; a locally-optimal permutation is then identified through a collection of local interchanges/rearrangements; the coordinates are re-estimated based on this identified permutation, and the whole process repeated until no change can be made in either the identified permutation or coordinate collection.

**The QA interchange/rearrangement heuristics**

The three m-functions that carry out general QA interchange/rearrangement heuristics all have the same general usage syntax (note the use of three dots to indicate a statement continuation in MATLAB):

\[
\begin{align*}
\text{[outperm,rawindex,allperms,index]} &= \text{pairwiseqa(prox,targ,inperm)} \\
\text{[outperm,rawindex,allperms,index]} &= \ldots \\
&\quad \text{rotateqa(prox,targ,inperm,kblock)} \\
\text{[outperm,rawindex,allperms,index]} &= \ldots \\
&\quad \text{insertqa(prox,targ,inperm,kblock)}
\end{align*}
\]
pairwiseqa.m carries out a QA maximization task using the pairwise interchanges of objects in the current permutation defining the row and column order of the data matrix. All possible such interchanges are generated and considered in turn, and whenever an increase in the cross-product index would result from a particular interchange, it is made immediately. The process continues until the current permutation cannot be improved upon by any such pairwise object interchange; this final locally optimal permutation is OUTPERM. The input beginning permutation is INPERM (a permutation of the first n integers); PROX is the $n \times n$ input proximity matrix and TARG is the $n \times n$ input target matrix (which are respective analogues of the matrices $A$ and $B$ of (1.5)); the final OUTPERM row and column permutation of PROX has the cross-product index RAWINDEX with respect to TARG. The cell array ALLPERMS contains INDEX entries corresponding to all the permutations identified in the optimization, from ALLPERMS{1} = INPERM to ALLPERMS{INDEX} = OUTPERM. (Note that within a MATLAB environment, entries of a cell array must be accessed through the curly braces, { }). rotateqa.m carries out a similar iterative QA maximization task but now uses the rotation (or inversion) of from 2 to KBLOCK (which is less than or equal to $n - 1$) consecutive objects in the current permutation defining the row and column order of the data matrix. insertqa.m relies on the reinsertion of from 1 to KBLOCK consecutive objects somewhere in the permutation defining the current row and column order of the data matrix.

The function uniscalqa.m

The function m-file, uniscalqa.m, carries out a unidimensional scaling of a symmetric dissimilarity matrix (with a zero main diagonal) using an iterative quadratic assignment strategy. We begin with an equally-spaced target, a (random) starting permutation, and use a sequential combination of the pairwise interchange/rotation/insertion heuristics; the target matrix is re-estimated based on the identified (locally optimal) permutation. The whole process is repeated until no changes can be made in the target or the identified (locally optimal) permutation. The explicit usage syntax is

$$[	ext{outperm}, 	ext{rawindex}, 	ext{allperms}, 	ext{index}, 	ext{coord}, 	ext{diff}] = \ldots$$

uniscalqa(prox, targ, inperm, kblock)

where all terms are (mostly) present in the three QA heuristic m-functions of the previous subsection. Here, COORD gives the final coordinates achieved, and DIFF provides the attained value for the least-squares loss function. A recording of a MATLAB session using number.dat follows; note the application of the built-in MATLAB function randperm(10) to obtain a random input permutation of the first 10 digits, and the use of the utility m-function, targlin.m (and the command targlin(10)), to generate a target matrix, targlinear, based on an equally (and unit) spaced set of coordinates. In the output given below, semicolons are placed after the invocation of the m-functions to initially suppress the output; transposes(‘) are then used on the output vectors to conserve space by only using row (as opposed to column) vectors in the listing.

```
>> load number.dat
```
1.3 Confirmatory and Nonmetric LUS

```matlab
>> targlinear = tarmlin(10);
>> inperm = randperm(10);
>> kblock = 2;
>> [outperm,rawindex,allperms,index,coord,diff] = ...
   uniscalqa(number,targlinear,inperm,kblock);

>> outperm

outperm =

    1    2    3    5    4    6    7    9   10    8

>> coord'

ans =

Columns 1 through 6

  -0.6570  -0.4247  -0.2608  -0.1492  -0.0566   0.0842

Columns 7 through 10

   0.1988   0.3258   0.4050   0.5345

>> diff

diff =

    1.9599
```

1.3 Confirmatory and Nonmetric LUS

In developing linear unidimensional scaling (as well as other types of) representations for a proximity matrix, it is convenient to have a general mechanism available for solving linear (in)equality constrained least-squares tasks. The two such instances discussed in this section involve (a) the confirmatory fitting of a given object order to a proximity matrix (through an m-file called `linfit.m`), and (b) the construction of an optimal monotonic transformation of a proximity matrix in relation to a given unidimensional ordering (through an m-file called `proxmon.m`). In both these cases, we rely on what can be called the Dykstra-Kaczmarz method. An equality constrained least-squares task may be rephrased as a linear system of equations, with the later solvable through a strategy of iterative projection as attributed to Kaczmarz (1937; see Bodewig, 1956, pp. 163–164): a more general inequality constrained least-squares task can also be approached through iterative projection as developed by Dykstra (1983). The Kaczmarz and Dykstra strategies...
are reviewed very briefly in the chapter addendum (Section 1.4), and implemented within the two m-files, `linfit.m` and `proxmon.m`, discussed below.

### 1.3.1 The confirmatory fitting of a given order using `linfit.m`

The m-function, `linfit.m`, fits a set of coordinates to a given proximity matrix based on some given input permutation, say, $\rho^{(0)}$. Specifically, we seek $x_1 \leq x_2 \leq \cdots \leq x_n$ such that $\sum_{i<j} (p_{\rho^{(0)}(i)\rho^{(0)}(j)} - |x_j - x_i|)^2$ is minimized (and where the permutation $\rho^{(0)}$ may not even put the matrix $\{p_{\rho^{(0)}(i)\rho^{(0)}(j)}\}$ into a monotonic form).

Using the syntax

```
[fit,diff,coord] = linfit(prox,inperm)
```

the matrix $\{|x_j - x_i|\}$ is referred to as the fitted matrix (FIT); COORD gives the ordered coordinates; and DIFF is the value of the least-squares criterion. The fitted matrix is found through the Dykstra-Kaczmarz method where the equality constraints defined by distances along a continuum are imposed to find the fitted matrix, i.e., if $i < j < k$, then $|x_i - x_j| + |x_j - x_k| = |x_i - x_k|$. Once found, the actual ordered coordinates are retrieved by the usual $t^{(\rho^{(0)})}_i$ formula used in (1.3) but computed on FIT.

The example below of the use of `linfit.m` fits two separate orders: the identity permutation and the one that we know is least-squares optimal (see Hubert, Arabie, and Meulman, 2002, for an explicit justification of optimality using a dynamic programming routine).

```matlab
>> load number.dat
>> inperm = [1 2 3 4 5 6 7 8 9 10];
>> [fit,diff,coord] = linfit(number,inperm);
>> coord'
```

```
ans =

Columns 1 through 6
-0.6570 -0.4247 -0.2608 -0.1392 -0.0666 0.0842

Columns 7 through 10
0.1988 0.3627 0.4058 0.4968
```

```matlab
>> diff

diff =

2.1046
```

```matlab
>> inperm = [1 2 3 5 4 6 7 9 10 8];
```
>> [fit,diff,coord] = linfit(number,inperm);
>> coord'

ans =

Columns 1 through 6
-0.6570  -0.4247  -0.2608  -0.1492  -0.0566  0.0842

Columns 7 through 10
0.1988   0.3258   0.4050   0.5345

>> diff
diff =

1.9599

1.3.2 The monotonic transformation of a proximity matrix using proxmon.m

The function, proxmon.m, provides a monotonically transformed proximity matrix that is closest in a least-squares sense to a given input matrix. The syntax is

[monproxpermut,vaf,diff] = proxmon(proxpermut,fitted)

Here, PROXPERMUT is the input proximity matrix (which may have been subjected to an initial row/column permutation, hence the suffix 'PERMUT') and FITTED is a given target matrix; the output matrix MONPROXPERMUT is closest to FITTED in a least-squares sense and obeys the order constraints obtained from each pair of entries in (the upper-triangular portion of) PROXPERMUT (and where the inequality constrained optimization is carried out using the Dykstra-Kaczmarz iterative projection strategy); VAF denotes 'variance-accounted-for' and indicates how much variance in MONPROXPERMUT can be accounted for by FITTED; finally, DIFF is the value of the least-squares loss function and is the sum of squared differences between the entries in FITTED and MONPROXPERMUT (actually, DIFF is one-half of such a sum because the loss function in (1.1) is over $i < j$).

In the notation of the previous section when fitting a given order, FITTED would correspond to the matrix $\{|x_j - x_i|\}$, where $x_1 \leq x_2 \leq \cdots \leq x_n$; the input PROXPERMUT would be $\{p_{\rho(i)\rho(j)}\}$; MONPROXPERMUT would be $\{f(p_{\rho(i)\rho(j)})\}$, where the function $f(\cdot)$ satisfies the monotonicity constraints, i.e., if $p_{\rho(i)\rho(j)} < p_{\rho(i')\rho(j)}$ for $1 \leq i < j \leq n$ and $1 \leq i' < j' \leq n$, then $f(p_{\rho(i)\rho(j)}) \leq f(p_{\rho(i')\rho(j')})$. The transformed proximity matrix $\{f(p_{\rho(i)\rho(j)})\}$ minimizes the least-squares criterion (DIFF) of

$$\sum_{i<j}(f(p_{\rho(i)\rho(j)}) - |x_j - x_i|)^2.$$
Chapter 1. Linear Unidimensional Scaling

over all functions \( f(\cdot) \) that satisfy the monotonicity constraints. The VAF is a normalization of this loss value by the sum of squared deviations of the transformed proximities from their mean:

\[
VAF = 1 - \frac{\sum_{i<j}(f(p_{\rho(i)\rho(j)}) - |x_j - x_i|)^2}{\sum_{i<j}(f(p_{\rho(i)\rho(j)}) - \bar{f})^2},
\]

where \( \bar{f} \) denotes the mean of the off-diagonal entries in \( \{f(p_{\rho(i)\rho(j)})\} \).

An application incorporating proxmon.m

The script m-file listed below gives an application of proxmon.m using the (globally optimal) permutation found previously for our number.dat matrix. First, linfit.m is invoked to obtain a fitted matrix (fit); proxmon.m then generates the monotonically transformed proximity matrix (monproxpermut) with VAF = .5821 and diff = 1.0623. The strategy is then repeated cyclically (i.e., finding a fitted matrix based on the monotonically transformed proximity matrix, finding a new monotonically transformed matrix, and so on). To avoid degeneracy (where all matrices would converge to zeros), the sum of squares of the fitted matrix is normalized; convergence is based on observing a minimal change (less than 1.0e-006) in the VAF. As indicated in the output below, the final VAF is .6672 with a diff of .9718. (Although the permutation found earlier for number.dat remains the same throughout the construction of the optimal monotonic transformation, in this particular example it would also remain optimal with the same VAF if the unidimensional scaling were repeated with monproxpermut now considered the input proximity matrix. Even though probably rare, other data sets might not have such an invariance, and it may be desirable to initiate an iterative routine that finds both a unidimensional scaling [i.e., an object ordering] in addition to monotonically transforming the proximity matrix.)

```matlab
>> load number.dat
inperm = [8 10 9 7 6 4 5 3 2 1];
[fit diff coord] = linfit(number,inperm);
[monproxpermut vaf diff] = ...
    proxmon(number(inperm,inperm),fit);
sumfitsq = sum(sum(fit.^2));
prevvaf = 2;
while (abs(prevvaf-vaf) >= 1.0e-006)
    prevvaf = vaf;
    [fit diff coord] = linfit(monproxpermut,1:10);
    sumnewfitsq = sum(sum(fit.^2));
    fit = sqrt(sumfitsq)*(fit/sqrt(sumnewfitsq));
    [monproxpermut vaf diff] = proxmon(number(inperm,inperm), fit);
end
fit
diff
```
1.3. Confirmatory and Nonmetric LUS

`coord`
`monoproxpermuit`
`vaf`

`fit =`

Columns 1 through 6

<p>| | | | | | |</p>
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<td>0.8974</td>
<td>0.8108</td>
<td>0.6649</td>
</tr>
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</table>

Columns 7 through 10

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<td>1.2231</td>
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<td>0.7872</td>
<td>1.1406</td>
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<tr>
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<td>0.3535</td>
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<tr>
<td>0.6397</td>
<td>0.4987</td>
<td>0.3535</td>
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</table>

diff =

0.9718

ans =

Columns 1 through 6

-0.4558 -0.3795 -0.3215 -0.1544 -0.0742 0.0609

Columns 7 through 10
### Chapter 1. Linear Unidimensional Scaling

\[
\begin{array}{cccccc}
0.0842 & 0.2147 & 0.3492 & 0.6764 & & \\
\end{array}
\]

\text{monproxperm}ut =

**Columns 1 through 6**

\[
\begin{array}{cccccc}
0 & 0.2612 & 0.2458 & 0.2612 & 0.2458 & 0.5116 \\
0.2612 & 0 & 0.2458 & 0.2458 & 0.4286 & 0.2458 \\
0.2458 & 0.2458 & 0 & 0.2458 & 0.5116 & 0.6899 \\
0.2612 & 0.2458 & 0.2458 & 0 & 0.2458 & 0.2458 \\
0.2458 & 0.4286 & 0.5116 & 0.2458 & 0 & 0.2612 \\
0.5116 & 0.2458 & 0.6899 & 0.2458 & 0.2612 & 0 \\
0.6080 & 0.5116 & 0.2458 & 0.2458 & 0.2458 & 0.2458 \\
0.6899 & 0.7264 & 0.2458 & 0.2612 & 0.5116 & 0.2458 \\
0.5116 & 0.5116 & 0.6899 & 0.6080 & 0.4286 & 0.2458 \\
1.2231 & 1.1406 & 1.0780 & 0.6899 & 0.7264 & 0.6080 \\
\end{array}
\]

**Columns 7 through 10**

\[
\begin{array}{cccccc}
0.6080 & 0.6899 & 0.5116 & 1.2231 & & \\
0.5116 & 0.7264 & 0.5116 & 1.1406 & & \\
0.2458 & 0.2458 & 0.6899 & 1.0780 & & \\
0.2458 & 0.2612 & 0.6080 & 0.6899 & & \\
0.2458 & 0.5116 & 0.4286 & 0.7264 & & \\
0.2458 & 0.2458 & 0.2458 & 0.6080 & & \\
0 & 0.1410 & 0.5116 & 0.6080 & & \\
0.1410 & 0 & 0.2458 & 0.4286 & & \\
0.5116 & 0.2458 & 0 & 0.2612 & & \\
0.6080 & 0.4286 & 0.2612 & 0 & & \\
\end{array}
\]

\text{vaf} =

\[
0.6672
\]
1.4 Appendix: The Dykstra-Kaczmarz Method for Solving Linear (In)equality Constrained Least-Squares Tasks

Kaczmarz’s method can be characterized as follows:

Given $A = \{a_{ij}\}$ of order $m \times n$, $x' = \{x_1, \ldots, x_n\}$, $b' = \{b_1, \ldots, b_m\}$, and assuming the linear system $Ax = b$ is consistent, define the set $C_i = \{x \mid a_{ij}x_j = b_i\}$, for $1 \leq i \leq m$. The projection of any $n \times 1$ vector $y$ onto $C_i$ is simply $y - (a_i'y - b_i)a_i(a_i'a_i)^{-1}$, where $a_i' = \{a_{i1}, \ldots, a_{in}\}$. Beginning with a vector $x_0$, and successively projecting $x_0$ onto $C_1$, and that result onto $C_2$, and so on, and cyclically and repeatedly reconsidering projections onto the sets $C_1, \ldots, C_m$, leads at convergence to a vector $x_0^*$ that is closest to $x_0$ (in vector 2-norm, so that $\sum_{i=1}^{n} (x_0_i - x_0^*)^2$ is minimized) and $Ax_0^* = b$. In short, Kaczmarz’s method iteratively solves least-squares tasks subject to equality restrictions.

Dykstra’s method can be characterized as follows:

Given $A = \{a_{ij}\}$ of order $m \times n$, $x_0' = \{x_{01}, \ldots, x_{0n}\}$, $b' = \{b_1, \ldots, b_m\}$, and $w' = \{w_1, \ldots, w_n\}$, where $w_j > 0$ for all $j$, find $x_0^*$ such that $a_i'x_0^* \leq b_i$ for $1 \leq i \leq m$ and $\sum_{i=1}^{n} w_i(x_0_i - x_0^*)^2$ is minimized. Again, (re)define the (closed convex) sets $C_i = \{x \mid a_{ij}x_j \leq b_i\}$ and when a vector $y \notin C_i$, its projection onto $C_i$ (in the metric defined by the weight vector $w$) is $y - (a_i'y - b_i)a_iW^{-1}(a_i'W^{-1}a_i)^{-1}$, where $W^{-1} = \text{diag}\{w_1^{-1}, \ldots, w_n^{-1}\}$. We again initialize the process with the vector $x_0$ and each set $C_1, \ldots, C_m$ is considered in turn. If the vector being carried forward to this point when $C_i$ is (re)considered does not satisfy the constraint defining $C_i$, a projection onto $C_i$ occurs. The sets $C_1, \ldots, C_m$ are cyclically and repeatedly considered but with one difference from the operation of Kaczmarz’s method — each time a constraint set $C_i$ is revisited, any changes from the previous time $C_i$ was reached are first “added back”. This last process ensures convergence to a (globally) optimal solution $x_0^*$ (see Dykstra, 1983). Thus, Dykstra’s method generalizes the equality restrictions that can be handled by Kaczmarz’s strategy to the use of inequality constraints.
Chapter 2

Linear Multidimensional Scaling

Chapter 1 gave an optimization strategy based on iterative quadratic assignment for the linear unidimensional scaling (LUS) task in the $L_2$-norm, with all implementations carried out within a MATLAB computational environment. The central LUS task involves arranging the $n$ objects in a set $S = \{O_1, O_2, \ldots, O_n\}$ along a single dimension, defined by coordinates $x_1, x_2, \ldots, x_n$, based on an $n \times n$ symmetric proximity matrix $P = \{p_{ij}\}$, whose (exclusively non-diagonal) non-negative entries are given a dissimilarity interpretation ($p_{ij} = p_{ji}$ for $1 \leq i, j \leq n$; $p_{ii} = 0$ for $1 \leq i \leq n$). The $L_2$ criterion
\[
\sum_{i<j} (p_{ij} - |x_j - x_i|)^2,
\] (2.1)
is minimized by the choice of the coordinates. The present chapter will give extensions to multidimensional scaling in the city-block metric (see Arabie, 1991, for a review of uses of this metric) for the $L_2$ norm. The computational routines to be discussed and illustrated are again freely available as MATLAB m-files. We also note that most of the references given in Chapter 1 would also be relevant here as background material on the basic LUS task, but that review will not be repeated. Also, we will not discuss (in this chapter) comparisons to other methods (or strategies) for multidimensional scaling in the city-block metric — for the development of some of these alternatives, see Brusco (2001), Brusco and Stahl (2004), Groenen, Heiser, and Meulman (1999), Hubert, Arabie, and Meulman (1997), and Hubert, Arabie, and Hesson-McInnis (1992).

In the extensions to city-block multidimensional scaling being pursued, a slight generalization to the basic unidimensional task that incorporates an additional additive constant will prove extremely convenient. So, in Section 2.1 we emphasize the more general least-squares loss function of the form
\[
\sum_{i<j} (p_{ij} - \{|x_j - x_i| - c\})^2,
\] (2.2)
where $c$ is some constant to be estimated along with the coordinates $x_1, \ldots, x_n$. Section 2.2 removes the restriction to fitting only a single unidimensional structure to
a symmetric proximity matrix, and relies on the type of computational approaches developed in Section 2.1 that include the augmentation by estimated additive constants. Based on these latter strategies, extensions are given to the use of multiple unidimensional structures through a procedure of successive residualization of the original proximity matrix (even though in this process, negative residuals are encountered and have to be fitted). For example, the fitting of two LUS structures to a proximity matrix \( \{p_{ij}\} \) could be rephrased as the minimization of an L2 loss function generalizing (2.2) to the form

\[
\sum_{i<j}(p_{ij} - [|x_{j1} - x_{i1}| - c_1] - [|x_{j2} - x_{i2}| - c_2])^2.
\]

(2.3)

The attempt to minimize (2.3) could proceed with the fitting of a single LUS structure to \( \{p_{ij}\} \), \(|x_{j1} - x_{i1}| - c_1\], and once obtained, fitting a second LUS structure, \(|x_{j2} - x_{i2}| - c_2\], to the residual matrix, \(\{p_{ij} - [|x_{j1} - x_{i1}| - c_1]\} \). The process would then cycle by repetitively fitting the residuals from the second linear structure by the first, and the residuals from the first linear structure by the second, until the sequence converges. In any case, obvious extensions would also exist to (2.3) for the inclusion of more than two LUS structures.

The explicit inclusion of two constants, \(c_1\) and \(c_2\), in (2.3) rather than adding these two together and including a single additive constant \(c\), deserves some additional introductory explanation. As would be the case in fitting a single LUS structure using the loss functions in (2.2), two interpretations exist for the role of the additive constant \(c\). We could consider \(|x_j - x_i|\} to be fitted to the translated proximities \(\{p_{ij} + c\}\], or alternatively, \(|x_j - x_i - c\} to be fitted to the original proximities \(\{p_{ij}\}\], where the constant \(c\) becomes part of the actual model. Although these two interpretations do not lead to any algorithmic differences in how we would proceed with minimizing the loss functions in (2.2), a consistent use of the second interpretation suggests that we frame extensions to the use of multiple LUS structures as we did in (2.3), where it is explicit that the constants \(c_1\) and \(c_2\) are part of the actual models to be fitted to the (untransformed) proximities \(\{p_{ij}\}\). Once \(c_1\) and \(c_2\) are obtained, they could be summed as \(c = c_1 + c_2\), and an interpretation made that we have attempted to fit a transformed set of proximities \(\{p_{ij} + c\}\) by the sum \(|x_{j1} - x_{i1}| + |x_{j2} - x_{i2}|\} (and in this latter case, a more usual terminology would be one of a two-dimensional scaling (MDS) based on the city-block distance function). However, such a further interpretation is unnecessary and could lead to at least some small terminological confusion in further extensions that we might wish to pursue. For instance, if some type of (optimal nonlinear) transformation, say \(f(\cdot)\), of the proximities is also sought (e.g., a monotonic function of some form as we do in Section 2.3), in addition to fitting multiple LUS structures, and where \(p_{ij}\) in (2.3) is replaced by \(f(p_{ij})\), and \(f(\cdot)\) is to be constructed, the first interpretation would require the use of a ‘doubly transformed’ set of proximities \(\{f(p_{ij}) + c\}\) to be fitted by the sum \(|x_{j1} - x_{i1}| + |x_{j2} - x_{i2}|\}. In general, it seems best to avoid the need to incorporate the notion of a double transformation in this context, and instead merely consider the constants \(c_1\) and \(c_2\) to be part of the models being fitted to a transformed set of proximities \(f(p_{ij})\).
2.1 The Incorporation of Additive Constants in LUS

In Section 2.1.1 below, we present and illustrate an m-function, linfitac.m, that fits in $L_2$ a given single unidimensional scale (by providing the coordinates $x_1, \ldots, x_n$) and the additive constant ($c$) for some fixed input object ordering along the continuum defined by a permutation $\rho^{(0)}$. This approach directly parallels the m-function given in the previous chapter called linfit.m, but now with an included additive constant estimation. The computational mechanisms implemented in linfitac.m are reviewed in Section 2.1.1.

2.1.1 The $L_2$ Fitting of a Single Unidimensional Scale (with an Additive Constant)

Given a fixed object permutation, $\rho^{(0)}$, we denote the set of all $n \times n$ matrices that are additive translations of the off-diagonal entries in the reordered symmetric proximity matrix $\{p_{\rho^{(0)}(i)\rho^{(0)}(j)}\}$ by $\Delta_{\rho^{(0)}}$, and let $\Xi$ be the set of all $n \times n$ matrices that represent the interpoint distances between all pairs of $n$ coordinate locations along a line. Explicitly,

$$\Delta_{\rho^{(0)}} \equiv \{\{q_{ij}\}\}, \text{ where } q_{ij} = p_{\rho^{(0)}(i)\rho^{(0)}(j)} + c, \text{ for some constant } c, i \neq j; q_{ii} = 0, \text{ for } 1 \leq i, j \leq n;$$

$$\Xi \equiv \{\{r_{ij}\}\}, \text{ where } r_{ij} = |x_j - x_i| \text{ for some set of } n \text{ coordinates, } x_1 \leq \cdots \leq x_n; \sum_i x_i = 0.$$

Alternatively, we could define $\Xi$ through a set of linear inequality (for non-negativity restrictions) and equality constraints (to represent the additive nature of distances along a line – as we did in linfit.m in the previous chapter). In either case, both $\Delta_{\rho^{(0)}}$ and $\Xi$ are closed convex sets (in a Hilbert space), and thus, given any $n \times n$ symmetric matrix with a zero main diagonal, its projection onto either $\Delta_{\rho^{(0)}}$ or $\Xi$ exists, i.e., there is a (unique) member of $\Delta_{\rho^{(0)}}$ or $\Xi$ at a closest (Euclidean) distance to the given matrix (e.g., see Cheney and Goldstein, 1959). Moreover, if a procedure of alternating projections onto $\Delta_{\rho^{(0)}}$ and $\Xi$ is carried out (where a given matrix is first projected onto one of the sets, and that result is then projected onto the second which result is in turn projected back onto the first, and so on), the process is convergent and generates members of $\Delta_{\rho^{(0)}}$ and $\Xi$ that are closest to each other (again, this last statement is justified in Cheney and Goldstein, 1959, Theorems 2 and 4).

Given any $n \times n$ symmetric matrix with a main diagonal of all zeros, which we denote arbitrarily as $U = \{u_{ij}\}$, its projection onto $\Delta_{\rho^{(0)}}$ may be obtained by a simple formula for the sought constant $c$. Explicitly, the minimum over $c$ of

$$\sum_{i<j} (p_{\rho^{(0)}(i)\rho^{(0)}(j)} + c - u_{ij})^2,$$

is obtained for

$$\hat{c} = (2/n(n-1)) \sum_{i<j} (u_{ij} - p_{\rho^{(0)}(i)\rho^{(0)}(j)}),$$
and thus, this last value defines a constant translation of the proximities necessary to generate that member of $\Delta_{\rho(0)}$ closest to $\mathbf{U} = \{u_{ij}\}$. For the second necessary projection and given any $n \times n$ symmetric matrix (again with a main diagonal of all zeros), that we denote arbitrarily as $\mathbf{V} = \{v_{ij}\}$ (but which in our applications will generally have the form $v_{ij} = p_{\rho(0)(i)\rho(0)(j)} + c$ for $i \neq j$ and some constant $c$), its projection onto $\Xi$ is somewhat more involved and requires minimizing

$$\sum_{i<j} (v_{ij} - r_{ij})^2,$$

over $r_{ij}$, where $\{r_{ij}\}$ is subject to the linear inequality non-negativity constraints, and the linear equality constraints of representing distances along a line (of the set $\Xi$). Although this is a (classic) quadratic programming problem for which a wide variety of optimization techniques has been published, we adopt (as we did in fitting a LUS without an additive constant in \texttt{linfit.m}), the Dykstra-Kaczmarz iterative projection strategy reviewed in the addendum (Section 1.4) to Chapter 1.

**The function linfitac.m**

As discussed above, the m-function, \texttt{linfitac.m}, fits a set of coordinates to a given proximity matrix based on some given input permutation, say, $\rho(0)$, plus an additive constant, $c$. The usage syntax of

$$[\text{fit}, \text{vaf}, \text{coord}, \text{addcon}] = \text{linfitac}(\text{prox}, \text{inperm})$$

is similar to that of \texttt{linfit.m} except for the inclusion (as output) of the additive constant \texttt{ADDCON}, and the replacement of the least-squares criterion of \texttt{DIFF} by the variance-accounted-for (\texttt{VAF}) given by the general formula

$$\text{vaf} = 1 - \frac{\sum_{i<j} (p_{\rho(0)(i)\rho(0)(j)} + c - |x_j - x_i|)^2}{\sum_{i<j} (p_{ij} - \bar{p})^2},$$

where $\bar{p}$ is the mean of the proximity values under consideration.

To illustrate the invariance of \texttt{VAF} to the use of linear transformations of the proximity matrix (although \texttt{COORD} and \texttt{ADDCON} obviously will change depending on the transformation used), we fitted the permutation found optimal to two different matrices: the original proximity matrix for \texttt{number.dat}, and one standardized to mean zero and variance one. The latter matrix is obtained with the utility \texttt{proxstd.m}, with usage explained in its m-file header comments given in Appendix A.

In the recording below (as well as earlier in Chapter 1), semicolons are placed after the invocation of the m-functions to suppress the output initially; transposes(\texttt{'}) are then used on the output vectors to conserve space by only using row (as opposed to column) vectors in the listing. Note that for the two proximity matrices employed, the \texttt{VAF} values are the same (.5612) but the coordinates and additive constants differ; a listing of the standardized proximity matrix is given in the output to show explicitly how negative proximities pose no problem for the fitting process that allows the incorporation of additive constants within the fitted model.
2.1. The Incorporation of Additive Constants in LUS

>> load number.dat
>> inperm = [1 2 3 5 4 6 7 9 10 8];
>> [fit,vaf,coord,addcon] = linfitac(number,inperm);
>> vaf
d
0.5612

>> coord'
d  
anse =

Columns 1 through 6
-0.3790  -0.2085  -0.1064  -0.0565  -0.0257  0.0533

Columns 7 through 10
0.1061  0.1714  0.1888  0.2565

>> addcon
d
addcon =
-0.3089

>> numberstan = proxstd(number,0.0)

numberstan =

Columns 1 through 6
0  -0.5919  0.2105  0.8258  0.7027  1.2934
-0.5919  0  -1.2663  -0.9611  0.5157  0.2302
0.2105  -1.2663  0  -0.9217  -2.3739  0.6387
0.8258  -0.9611  -0.9217  0  -0.6313  -0.5525
0.7027  0.5157  -2.3739  -0.6313  0  -0.6510
1.2934  0.2302  0.6387  -0.5525  -0.6510  0
1.2147  1.0670  -0.5919  -1.1876  -0.7544  -0.7150
1.8103  0.4369  1.2541  0.2498  0.9882  -0.6953
1.3771  1.2294  -0.8577  1.2934  -1.4534  0.6387
1.5199  0.4123  1.3131  -1.3697  0.6978  0.2498

Columns 7 through 10
1.2147  1.8103  1.3771  1.5199
Chapter 2. Linear Multidimensional Scaling

\[
\begin{array}{cccc}
1.0670 & 0.4369 & 1.2294 & 0.4123 \\
-0.5919 & 1.2541 & -0.8577 & 1.3131 \\
-1.1876 & 0.2498 & 1.2934 & -1.3697 \\
-0.7544 & 0.9882 & -1.4534 & 0.6978 \\
-0.7150 & -0.6953 & 0.6387 & 0.2498 \\
0 & -0.6116 & -0.9414 & -1.2072 \\
-0.6116 & 0 & -0.6953 & -0.4049 \\
-0.9414 & -0.6953 & 0 & -0.7347 \\
-1.2072 & -0.4049 & -0.7347 & 0 \\
\end{array}
\]

\[
>> \ [\text{fit},vaf,coord,addcon] = \text{linfitac(numberstan,inperm)}; \\
>> vaf \\
\begin{array}{c}
vaf = \\
0.5612 \\
\end{array}
\]

\[
>> \text{coord}' \\
\begin{array}{c}
\text{ans} = \\
\text{Columns 1 through 6} \\
-1.8656 & -1.0262 & -0.5235 & -0.2783 & -0.1266 & 0.2624 \\
\text{Columns 7 through 10} \\
0.5224 & 0.8435 & 0.9292 & 1.2626 \\
\end{array}
\]

\[
>> \text{addcon} \\
\begin{array}{c}
\text{addcon} = \\
1.1437 \\
\end{array}
\]

2.2 Finding and Fitting Multiple Unidimensional Scales

As reviewed in this chapter's introduction, the fitting of multiple unidimensional structures will be done by (repetitive) successive residualization, along with a reliance on the m-function, \texttt{linfitac.m}, to fit each separate unidimensional structure, including its associated additive constant. The m-function for this two-dimensional scaling, \texttt{biscalqa.m}, is a bidimensional strategy for the \(L_2\) loss function of (2.3). It has the syntax
2.2. Finding and Fitting Multiple Unidimensional Scales

[outpermone, outpermtwo, coordone, coordtwo, fitone, fittwo, ...  
addconone, addcontwo, vaf] = biscalqa(prox,...  
targone, targtwo, inpermone, inpermtwo, kblock, nopt)

where the variables are similar to linfitac.m, but with a suffix of ONE or TWO to indicate which one of the two unidimensional structures is being referenced. The new variable NOPT controls the confirmatory or exploratory fitting of the two unidimensional scales; a value of NOPT = 0 will fit the two scales indicated by INPERMONE and INPERMTWO in a confirmatory manner; if NOPT = 1, iterative quadratic assignment (QA) is used to locate the better permutations to fit.

In the example given below, the input PROX is the standardized (to a mean of zero and a standard deviation of one) 10×10 proximity matrix based on number.dat (referred to as STANNUMBER); TARGONE and TARGTWO are identical 10×10 equally-spaced target matrices; INPERMONE and INPERMTWO are different random permutations of the first 10 integers; KBLOCK is set at 2 (for the iterative QA subroutines). In the output, OUTPERMONE and OUTPERMTWO refer to the object orders; COORDONE and COORDTWO give the coordinates; FITONE and FITTWO are based on the absolute coordinate differences for the two unidimensional structures; ADDCONONE and ADDCONTWO are the two associated additive constraints; and finally, VAF is the variance-accounted-for in PROX by the two-dimensional structure.

```matlab
>> load number.dat
>> stannumber = proxstd(number,0.0);
>> inpermone = randperm(10);
>> inpermtwo = randperm(10);
>> kblock = 2;
>> nopt = 1;
>> targone = targlin(10);
>> targtwo = targone;
>> [outpermone, outpermtwo, coordone, coordtwo, fitone, fittwo,...
  addconone, addcontwo, vaf] = biscalqa(stannumber, targone,...
  targtwo, inpermone, inpermtwo, kblock, nopt);

>> outpermone
outpermone =
    10   8    9    7    6    5    4    3    2    1

>> outpermtwo
outpermtwo =
    6    8    2   10    4    7    1    3    5    9

>> coordone'
```
ans =

Columns 1 through 6
-1.4191 -1.0310 -1.0310 -0.6805 -0.0858 -0.0009

Columns 7 through 10
0.2915 0.5418 1.2363 2.1786

>> coordtwo'

ans =

Columns 1 through 6
-1.1688 -0.9885 -0.3639 -0.2472 -0.2472 0.1151

Columns 7 through 10
0.2629 0.8791 0.8791 0.8791

>> addconone

addconone =

1.3137

>> addcontwo

addcontwo =

0.8803

>> vaf

vaf =

0.8243

Although we have used the proximity matrix in number.dat primarily as a convenient numerical example to illustrate our various m-functions, the substantive interpretation for this particular two-dimensional structure is rather remarkable and worth noting. The first dimension reflects number magnitude perfectly (in its coordinate order) with two objects (the actual digits 7 8) at the same (tied) coordinate value. The second axis reflects the structural characteristics perfectly,
2.3. Incorporating Monotonic Transformation of a Proximity Matrix

with the coordinates split into the odd and even numbers (the digits 6 0 2 4 8 in the second five positions; 5 7 1 9 3 in the first five); there is a grouping of 2 4 8 at the same coordinates (reflecting powers of 2); a grouping of 9 3 6 (reflecting multiples of three) and of 9 3 at the same coordinates (reflecting the powers of 3); the odd numbers 7 5 that are not powers of 3 are at the extreme two coordinates of this second dimension.

We will not explicitly illustrate its use here, but a tridimensional m-function, triscalqa.m, is an obvious generalization of biscalqa.m. Also, the pattern of programming shown could be used directly as a pattern for extensions beyond three unidimensional structures.

2.3 Incorporating Monotonic Transformation of a Proximity Matrix

As a direct extension of the m-function, biscalqa.m, discussed in the last section, the file bimonscalqa.m provides an optimal monotonic transformation (by incorporating the use of proxmon.m discussed in Chapter 1) of the original proximity matrix given as input in addition to the later’s bidimensional scaling. To prevent degeneracy, the sum-of-squares value for the initial input proximity matrix is maintained in the optimally transformed proximities; the overall strategy is iterative with termination dependent on a change in the VAF being less than 1.0e-005. The usage syntax is almost identical to that of biscalqa.m except for the inclusion of the monotonically transformed proximity matrix MONPROX as an output matrix:

\[
[\ldots\text{monprox}] = \text{bimonscalqa}(\ldots)
\]

The ellipses directly above indicate that the same items should be used as in biscalqa.m. If bimonscalqa.m had been used in the numerical example of the previous section, the same results given would have been output initially plus the results for the optimally transformed proximity matrix. We give this additional output below, which shows that the incorporation of an optimal monotonic transformation provides an increase in the VAF from .8243 to .9362; the orderings on the two dimensions remain the same as well as the nice substantive explanation of the previous section.

\[
\text{outpermone}
\]

\[
\text{outpermone} =
\begin{array}{cccccccc}
10 & 8 & 9 & 7 & 6 & 5 & 4 & 3 & 2 & 1
\end{array}
\]

\[
\text{outpermtwo}
\]

\[
\text{outpermtwo} =
\begin{array}{cccccccc}
6 & 8 & 2 & 4 & 10 & 7 & 1 & 3 & 5 & 9
\end{array}
\]
Chapter 2. Linear Multidimensional Scaling

>> coordone'
ans =
Columns 1 through 7
-1.6247  -1.1342  -1.1342  -0.5857  -0.1216  -0.0775  0.3565
Columns 8 through 10
0.6409  1.3290  2.3514

>> coordtwo'
ans =
Columns 1 through 7
-1.0035  -0.8467  -0.3480  -0.3242  -0.3242  0.1196  0.3891
Columns 8 through 10
0.7793  0.7793  0.7793

>> addconone
addconone =
1.4394

>> addcontwo
addcontwo =
0.7922

>> vaf
vaf =
0.9362

>> monprox
monprox =
### 2.4 Confirmatory Extensions to City-Block Individual Differences Scaling

Columns 1 through 7

\[
\begin{array}{ccccccc}
0 & -0.7387 & -0.1667 & 0.5067 & 0.5067 & 1.4791 & 1.0321 \\
-0.7387 & 0 & -0.8218 & -0.8218 & 0.5067 & -0.1667 & 0.5067 \\
-0.1667 & -0.8218 & 0 & -0.8218 & -1.6174 & 0.5067 & -0.7387 \\
0.5067 & -0.8218 & -0.8218 & 0 & -0.7387 & -0.7387 & 0.5067 \\
0.5067 & 0.5067 & -1.6174 & -0.7387 & 0 & -0.7387 & -0.8218 \\
1.4791 & -0.1667 & 0.5067 & -0.7387 & -0.7387 & 0 & -0.8218 \\
1.0321 & 0.5067 & -0.7387 & -0.8218 & -0.8218 & -0.8218 & 0 \\
2.6590 & 0.5067 & 1.0321 & -0.1667 & 0.5067 & -0.8218 & -0.7387 \\
1.7609 & 1.0321 & -0.8218 & 1.0321 & -1.2541 & 0.5067 & -0.8218 \\
2.6231 & 0.5067 & 1.4791 & -0.8218 & 0.5067 & -0.0534 & -0.8218 \\
\end{array}
\]

Columns 8 through 10

\[
\begin{array}{cccc}
2.6590 & 1.7609 & 2.6231 \\
0.5067 & 1.0321 & 0.5067 \\
1.0321 & -0.8218 & 1.4791 \\
-0.1667 & 1.0321 & -0.8218 \\
0.5067 & -1.2541 & 0.5067 \\
-0.8218 & 0.5067 & -0.0534 \\
-0.7387 & -0.8218 & -0.8218 \\
0 & -0.7387 & -0.7387 \\
-0.7387 & 0 & -0.8218 \\
-0.7387 & -0.8218 & 0 \\
\end{array}
\]

Although we will not provide an example of its use here, trimonscalqa.m extends triscalqa.m to include an optimal monotonic transformation of whatever is given as the original input proximity matrix.

### 2.4 Confirmatory Extensions to City-Block Individual Differences Scaling

An obvious conclusion to this chapter is that if one is interested in (nonmetric) city-block scaling in two or three dimensions within $L_2$, the routines referred to in two dimensions as biscalqa.m and bimonscalqa.m; or in three dimensions as triscalqa.m and trimonscalqa.m, would be natural alternatives to consider. One aspect of all of these given m-files that we have not emphasized but will in this chapter’s concluding comments, is their possible usage in the confirmatory context (by setting the NOPT switch to 0), and fitting various fixed object orderings in multiple dimensions. One possible application of this type of confirmatory fitting would be in an individual differences scaling context. Explicitly, we begin with a collection of, say, $K$ proximity matrices, $P_1, \ldots, P_K$, obtained from $K$ separate sources, and through some weighting and averaging process construct a single aggregate proximity matrix, $P_A$. On the basis of $P_A$, suppose a two-dimensional city-block scaling is constructed (using, say, biscalqa.m); we label the latter the “common space”
consistent with what is usually done in the weighted Euclidean model (e.g., see the INDSCAL model of Carroll and Chang [1970, Arabie, Carroll, & DeSarbo, 1987; Carroll & Arabie, 1988], or the PROXSCAL program in the Categories Module of SPSS — Busing, Commandeur, and Heiser, 1997). Each of the $K$ proximity matrices then can be used in a confirmatory fitting of the object orders along the two axes. Thus, a very general “subject/private space” is generated for each source and where the actual coordinates along both axes are unique to that source, subject only to the object order constraints of the group space. This strategy provides an individual differences model generalization over the usual weighted Euclidean model where the latter allows only differential axes scaling (stretching or shrinking) in generating the private spaces. These kinds of individual difference generalizations exist both for multiple unidimensional scalings in $L_2$ as well as for other types of proximity matrix representations such as ultrametrics and additive trees (given in Parts II and III).
Chapter 3

Circular Scaling

This chapter will discuss circular unidimensional scaling (CUS), where the objective is to place the \( n \) objects around a closed continuum such that the reconstructed distance between each pair of objects, defined by the minimum length over the two possible paths that join the objects, reflects the given proximities as well as possible. Explicitly, and in analogy with the loss function for linear unidimensional scaling (LUS) in (2.2), we wish to find a set of coordinates, \( x_1, \ldots, x_n \), and an \((n + 1)\)th value, \( x_0 \geq |x_j - x_i| \) for all \( 1 \leq i \neq j \leq n \), minimizing

\[
\sum_{i<j}(p_{ij} + c - \min\{|x_j - x_i|, x_0 - |x_j - x_i|\})^2,
\]  

(3.1)

or equivalently,

\[
\sum_{i<j}(p_{ij} - [\min\{|x_j - x_i|, x_0 - |x_j - x_i|\} - c])^2,
\]  

(3.2)

where \( c \) is again some constant to be estimated. The value \( x_0 \) represents the total length of the closed continuum, and the expression, \( \min\{|x_j - x_i|, x_0 - |x_j - x_i|\} \), gives the minimum length over the two possible paths joining objects \( O_i \) and \( O_j \).

In theory, the CUS task could again be solved by complete enumeration of the loss function in (3.1) over a finite but also typically enormous set for even moderate \( n \). Here, we have all possible distinct object orderings around a closed continuum, \((n - 1)!/2\), as well as for each such ordering, all the possible inflection patterns for where the directionality of the minimum distance calculation changes for the object pairs. Obviously, for general use and analogous to LUS, some type of search strategy is called for to generate the better orderings and inflection patterns around a closed continuum but with feasible computational effort.

The current chapter is organized around the optimization problems posed by CUS and the two main subtasks of obtaining an appropriate object ordering around a closed continuum, and the additional relative placement of the objects according to the directionality of minimum distance calculations. Once these orderings and
relative placements have been identified, the estimation of the additive constant in both loss functions (3.1) and (3.2), as well as the identification of the actual coordinates, proceeds through a process of alternating projections onto two closed convex sets. One is defined by the set of all translations of the original proximity values by a constant; the second closed convex set is characterized by the collection of all coordinate structures consistent with the given object ordering and the relative object placement that provides the directionality of minimum distance calculations around the closed continuum. In the specific contexts we consider, the process of alternating projections is convergent (see Cheney & Goldstein, 1956), and generates a final solution defined by two points within the two convex sets that are closest to each other. Although the emphasis in the present chapter is on a single symmetric proximity matrix defined for one set of objects to be fitted by a CUS model, Section 3.2 will also develop a fairly direct extension to the use of sums of such circular structures that can be fitted to a single proximity matrix through successive residualizations of the given matrix (this strategy is analogous to what was done in Chapter 2 for multiple LUS representations).

As a final preliminary note on the data that will be used in this chapter for numerical illustrations, a rather well-known proximity matrix is given in Table 3.1 (and called ‘morse_digits.dat’). The later is a $10 \times 10$ proximity matrix for the ten Morse Code symbols that represent the first ten digits: (0: $\text{−−−−−}$; 1: $\text{•−−−−}$; 2: $\text{••−−−}$; 3: $\text{•••−−}$; 4: $\text{••••−}$; 5: $\text{•••••}$; 6: $\text{−••••}$; 7: $\text{−−•••}$; 8: $\text{−−−••}$; 9: $\text{−−−−•}$). The entries in Table 3.1 have a dissimilarity interpretation and are defined for each object pair by $2.0$ minus the sum of the two proportions for a group of subjects used by Rothkopf (1957) representing “same” judgments to the two symbols when given in the two possible presentation orders of the signals. Based on previous multidimensional scalings of the complete data set involving all of the Morse code symbols and in which the data of Table 3.1 are embedded (e.g., see Shepard, 1963; Kruskal & Wish, 1978), it might be expected that the symbols for the digits would form a clear linear unidimensional structure that would be interpretable according to a regular progression in the number of dots to dashes. It turns out, as discussed in greater detail below, that a circular model (or actually, a sum of circular structures) is probably more consistent with the patterning of the proximities in Table 3.1 than are representations based on linear unidimensional scalings. (For completeness, we might note that the possible usefulness of a circular model for these specific proximities [but not to the consideration of multiple circular structures] has been pointed out before in the literature, most notably in the two-dimensional metric scaling given as an illustrative example by Mardia, Kent, and Bibby, 1979, p. 404, or in the combinatorial optimization approach of quadratic assignment discussed by Hubert and Schultz, 1976, which is based on permuting the rows and columns of a proximity matrix to achieve a best fit against a fixed target matrix.)
3.1 The Mechanics of CUS

The CUS task as characterized by the loss function in (3.1) can be considered in two stages as with the presentation of LUS. One subtask is the identification of an ordering of the \( n \) objects around a closed continuum that again will be denoted by a permutation of the first \( n \) integers, \( \phi(\cdot) \), such that \( \phi(i) = j \) if object \( O_j \) is placed at position \( i \); here, position 1 is arbitrarily specified at some point along the closed continuum, and the order of the positions from 1 to \( n \) is, for convenience, taken clockwise. In addition to \( \phi(\cdot) \), a set of inflection points must be identified for the \( n \) positions to indicate where the minimum distance calculation must change direction around the closed continuum. Explicitly, a set of \( n-1 \) integers, \( 1 \leq k_1 \leq \cdots \leq k_{n-1} \leq n \), is sought, where \( k_i \) is associated with position \( i \), \( 1 \leq i \leq n-1 \). For positions \( i < j \), the minimum distance is in the clockwise direction when \( j \leq k_i \), and in the counterclockwise direction when \( j > k_i \) (we note that an integer \( k_n \) for position \( n \) is unnecessary, and any \( k_i \) equal to \( n \) merely indicates that for all positions \( j \), for \( i < j \), the minimum distance is always in the clockwise direction). The second subtask, once given \( \phi(\cdot) \) and \( k_1, \ldots, k_{n-1} \), is the estimation of the set of coordinates and the additive constant \( c \) to fit the proximities. We again discuss these two subtasks in the reverse order.

3.1.1 The Estimation of \( c \) and \( \min \{|x_j - x_i|, x_0 - |x_j - x_i|\} \) for a Fixed Permutation and Set of Inflection Points

For notational convenience, the set of all \( n \times n \) matrices that are additive translations of the off-diagonal entries in the reordered proximity matrix, \( \{p_{\phi(i)\phi(j)}\} \), will again be denoted by \( \Delta_{\phi} \) (see Section 2.1.1); the set of all \( n \times n \) matrices that represent the distances around the closed continuum based on the inflection points \( k_1, \ldots, k_{n-1} \) will be more fully denoted by \( \Xi(k_1, \ldots, k_{n-1}) \) and explicitly defined as follows:

\[
\Xi(k_1, \ldots, k_{n-1}) \equiv \{\{r_{ij}\}\}, \text{ where } r_{ij} = |x_j - x_i| \text{ for } i < j \leq k_i; \quad r_{ij} = x_0 -
\]

Table 3.1. A proximity matrix, morse_digits.dat, for the ten Morse code symbols representing the first ten digits (data from Rothkopf, 1957)

<table>
<thead>
<tr>
<th></th>
<th>0.00</th>
<th>0.75</th>
<th>1.69</th>
<th>1.87</th>
<th>1.76</th>
<th>1.77</th>
<th>1.59</th>
<th>1.26</th>
<th>0.86</th>
<th>0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
<td>0.00</td>
<td>0.82</td>
<td>1.54</td>
<td>1.85</td>
<td>1.72</td>
<td>1.51</td>
<td>1.50</td>
<td>1.45</td>
<td>1.63</td>
<td></td>
</tr>
<tr>
<td>1.69</td>
<td>0.82</td>
<td>0.00</td>
<td>1.25</td>
<td>1.47</td>
<td>1.33</td>
<td>1.66</td>
<td>1.57</td>
<td>1.83</td>
<td>1.81</td>
<td></td>
</tr>
<tr>
<td>1.87</td>
<td>1.54</td>
<td>1.25</td>
<td>0.00</td>
<td>0.89</td>
<td>1.32</td>
<td>1.53</td>
<td>1.74</td>
<td>1.85</td>
<td>1.86</td>
<td></td>
</tr>
<tr>
<td>1.76</td>
<td>1.85</td>
<td>1.47</td>
<td>0.89</td>
<td>0.00</td>
<td>1.41</td>
<td>1.64</td>
<td>1.81</td>
<td>1.90</td>
<td>1.90</td>
<td></td>
</tr>
<tr>
<td>1.77</td>
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<td>1.33</td>
<td>1.32</td>
<td>1.41</td>
<td>0.00</td>
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<td>1.84</td>
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<td>0.70</td>
<td>0.00</td>
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<td>1.38</td>
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<td>1.85</td>
<td>1.90</td>
<td>1.84</td>
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<tr>
<td>0.95</td>
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<td>1.81</td>
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<td>1.90</td>
<td>1.64</td>
<td>1.70</td>
<td>1.22</td>
<td>0.41</td>
<td>0.00</td>
<td></td>
</tr>
</tbody>
</table>
\[ |x_j - x_i| \text{ for } i < j \text{ and } j > k_i; \ r_{ji} = r_{ij} \text{ for } 1 \leq i < j \leq n; \ r_{ii} = 0 \text{ for } 1 \leq i \leq n; \]

for some collection of coordinates, \( x_1, \ldots, x_n \), and an \((n + 1)^{st}\) value, \( x_0 \), where \( x_1 \leq \cdots \leq x_n \leq x_0; \ x_1 = 0.0; \) and \( |x_j - x_i| \leq x_0 - |x_j - x_i| \) for \( i < j \leq k_i; \ |x_j - x_i| \geq x_0 - |x_j - x_i| \) for \( i < j, \) and \( j > k_i. \)

As noted in this definition, the first position, \( x_1 \), is specified without loss of generality to be 0.0; the value, \( x_0 \), can either be interpreted as the length of the closed continuum or as a second coordinate value attached to the first position but taken in the clockwise direction. Given \( \Delta_{x} \) and \( \Xi(k_1, \ldots, k_{n-1}) \) (where the latter can be defined through a set of linear inequality/equality constraints), the process of alternating projections onto \( \Delta_{x} \) and \( \Xi(k_1, \ldots, k_{n-1}) \) would proceed exactly as in LUS.

### 3.1.2 Obtaining Object Orderings and Inflection Points Around a Closed Continuum

Identifying an object ordering around a closed continuum to be used in the minimization of the loss function in (3.1) follows the same pattern as for LUS. The cross-product statistic in (1.5) is again maximized but with a different \( n \times n \) (target) matrix, \( T = \{t_{ij}\} \), initially defined by \( n \) positions equally-spaced around a closed continuum, i.e., \( t_{ij} = \min\{|i - j|, n - |i - j|\} \) for \( 1 \leq i, j \leq n \) (as in LUS, this target could eventually be replaced, now by \( t_{ij} = \min\{|x_j - x_i|, x_0 - |x_j - x_i|\} \) based on the outcome of the minimization of (3.1)). Given some best permutation, \( \varphi_{K(i)} \), obtained through the initial target and set of local operations on some randomly given initial permutation, a collection of inflection points, \( k_1, \ldots, k_{n-1} \), still must be generated before the optimization of (3.1) can continue. This latter task will be approached through a heuristic application of an iterative projection strategy of the same general type developed by Hubert and Arabie (1995b) for the fitting of various graph-theoretic structures to a symmetric proximity matrix.

To attempt an identification of \( k_1, \ldots, k_{n-1} \) given the permutation \( \varphi_{K(i)} \), we begin with the reordered proximity matrix \( \{p_{\varphi_{K(i)}}\} \), and initialize a process of iterative projection onto the class of constraints given by the structure \( \Xi(k_1, \ldots, k_{n-1}) \) but with one exception necessitated by the fact that an appropriate set of values for \( k_1, \ldots, k_{n-1} \) is not yet known. Explicitly, when considering a pair of positions, \( i < j \) (\( 2 \leq j - i \)), and the two possible constraints that could be imposed, i.e., either \( r_{i(i+1)} + \cdots + r_{(j-1)j} = r_{ij} \) or \( r - (r_{i(i+1)} + \cdots + r_{(j-1)j}) = r_{ij} \) for \( r = r_{12} + \cdots + r_{(n-1)n} + r_{1n} \), we select according to which left side is smaller, based on the current entries in the matrix being carried forward to this point, and impose that specific constraint. Otherwise, the process continues cyclically through the whole set of constraints, and for each time a constraint is reconsidered, any changes that were made the previous time the constraint was encountered are first “undone”.

Because of the procedure of redressing the (immediately) previous changes each time a constraint is reconsidered, the process just described may not converge and could eventually oscillate through a finite collection of distinct matrices. If such nonconvergence is observed, and previous changes from that point on
3.1. The Mechanics of CUS

are not redressed, the process will then converge to a matrix in $\Xi(k_1,\ldots,k_{n-1})$ for some specific values of $k_1,\ldots,k_{n-1}$. A justification for this last assertion of convergence is given by the general results presented in Hubert and Arabie (1995b); also, that source provides empirical evidence that as a heuristic optimization strategy, it is generally better to begin with the procedure of redressing previous changes until an oscillation is observed, rather than immediately starting without the process of redressing previous changes (which would also produce a matrix in $\Xi(k_1,\ldots,k_{n-1})$ for some specific $k_1,\ldots,k_{n-1}$). It should also be noted that although convergence to some matrix in $\Xi(k_1,\ldots,k_{n-1})$ is guaranteed by the strategy just described, and thus to an identified fixed collection of inflection points, $k_1,\ldots,k_{n-1}$, the latter matrix may now not be optimal for this collection of inflection points in the minimization of (3.1). Specifically, even though the identification of the collection $k_1,\ldots,k_{n-1}$ can proceed by a process of iterative projection and an updating of a matrix $\{r_{ij}\}$ to produce a member of $\Xi(k_1,\ldots,k_{n-1})$, because of the possible nonconvergence noted above and the subsequent lack of redressing previous changes from that point on, the matrix identified in $\Xi(k_1,\ldots,k_{n-1})$ may not be the best achievable even for this particular collection of inflection points (although in our computational experience it is typically very close to being optimal). Thus, as a “polishing” step to ensure that an optimal member of $\Xi(k_1,\ldots,k_{n-1})$ is identified, the collection $k_1,\ldots,k_{n-1}$ and the permutation $\varphi_K(\cdot)$ should be used anew in the optimization of (3.1) to obtain the optimal target matrix, 
\[
\min\{ |x_j - x_i|, x_0 - |x_j - x_i| \}
\].

3.1.3 The Circular Unidimensional Scaling Utilities, cirfit.m and cirfitac.m

The two circular unidimensional scaling utilities, that implement the mechanics of fitting the CUS model (including the identification of inflection points), parallel the LUS utilities of linfit.m and linfitac.m. The m-file cirfit.m does a confirmatory fitting of a given order (assumed to be an object ordering around a closed unidimensional structure) using the Dykstra-Kaczmarz iterative projection least-squares method. The usage syntax is cirfit.m and cirfitac.m,

\[
[\text{fit}, \text{diff}] = \text{cirfit}(\text{prox}, \text{inperm})
\]

where \text{INPERM} is the given order; \text{FIT} is an $n \times n$ matrix fitted to PROX(INPERM, INPERM) with a least-squares value \text{DIFF}. The syntax for the routine, cirfitac.m, is the same except for the inclusion of an additive constant, ADDCON, and the use of VAF rather than DIFF:

\[
[\text{fit}, \text{vaf}, \text{addcon}] = \text{cirfitac}(\text{prox}, \text{inperm})
\]

In brief, then, the type of matrix being fitted to the proximity matrix has the form

\[
\min\{ |x_{\rho(j)} - x_{\rho(i)}|, x_0 - |x_{\rho(j)} - x_{\rho(i)}| \} - c,
\]

where $c$ is an estimated additive constant (assumed equal to zero in cirfit.m), $x_{\rho(1)} \leq x_{\rho(2)} \leq \cdots \leq x_{\rho(n)} \leq x_0$, and the last coordinate, $x_0$, is the circumference

\[
\].
of the circular structure. We can obtain these latter coordinates from the adjacent spacings in the output matrix \textit{FIT}.

As an example, we applied \texttt{cirfit.m} to the \texttt{morse_digits} proximity matrix with an assumed identity input permutation; the spacings around the circular structure between the placements for objects 1 and 2 is .5337; 2 and 3: .7534; 3 and 4: .6174; 4 and 5: .1840; 5 and 6: .5747; 6 and 7: .5167; 7 and 8: .3920; 8 and 9: .5467; 9 and 10: .1090; and back around between 10 and 1: .5594 (the sum of all these adjacent spacings is 4.787 and is the circumference \((x_0)\) of the circular structure).

For \texttt{cirfitac.m} the additive constant was estimated as -.8031 with a VAF of .7051; here, the spacings around the circular structure between the placements for objects 1 and 2 is .2928; 2 and 3: .4322; 3 and 4: .2962; 4 and 5: .0234; 5 and 6: .3338; 6 and 7: .2758; 7 and 8: .2314; 8 and 9: .2800; 9 and 10: .0000; and back around between 10 and 1: .2124 (here, \(x_0\) has a value of 2.378).

\begin{verbatim}
>> load morse_digits.dat
>> [fit,diff] = cirfit(morse_digits,1:10)

fit =

Columns 1 through 5

<table>
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<tr>
<th></th>
<th>0</th>
<th>0.5337</th>
<th>1.2871</th>
<th>1.9044</th>
<th>2.0884</th>
</tr>
</thead>
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<td>1.3707</td>
<td>1.5547</td>
<td></td>
</tr>
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</tr>
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<td>0.6174</td>
<td>0</td>
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<td></td>
</tr>
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<td>0.1840</td>
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</tr>
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<td>2.1294</td>
<td>1.3761</td>
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</tr>
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<td>1.8927</td>
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</tr>
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<td>2.2847</td>
<td>1.6674</td>
<td>1.4834</td>
<td></td>
</tr>
<tr>
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<td>1.2021</td>
<td>1.9554</td>
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<td>2.0301</td>
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<tr>
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<td>1.8464</td>
<td>2.3231</td>
<td>2.1391</td>
<td></td>
</tr>
</tbody>
</table>

Columns 6 through 10

| 2.1237 | 1.6071 | 1.2151 | 0.6684 | 0.5594 |
| 2.1294 | 2.1407 | 1.7487 | 1.2021 | 1.0931 |
| 1.3761 | 1.8927 | 2.2847 | 1.9554 | 1.8464 |
| 0.7587 | 1.2754 | 1.6674 | 2.2141 | 2.3231 |
| 0.5747 | 1.0914 | 1.4834 | 2.0301 | 2.1391 |
| 0     | 0.5167 | 0.9087 | 1.4554 | 1.5644 |
| 0.5167 | 0     | 0.3920 | 0.9387 | 1.0477 |
| 0.9087 | 0.3920 | 0     | 0.5467 | 0.6557 |
| 1.4554 | 0.9387 | 0.5467 | 0      | 0.1090 |
| 1.5644 | 1.0477 | 0.6557 | 0.1090 | 0      |
\end{verbatim}
3.1. The Mechanics of CUS

\[
\text{diff} = 7.3898
\]

\[
\text{>> [fit,vaf,addcon] = cirfitac(morse_digits,1:10)}
\]

\[
\text{fit} =
\begin{align*}
\text{Columns 1 through 5} \\
&0 \ 0.2928 \ 0.7250 \ 1.0212 \ 1.0446 \\
&0.2928 \ 0 \ 0.4322 \ 0.7284 \ 0.7518 \\
&0.7250 \ 0.4322 \ 0 \ 0.2962 \ 0.3196 \\
&1.0212 \ 0.7284 \ 0.2962 \ 0 \ 0.0234 \\
&1.0446 \ 0.7518 \ 0.3196 \ 0.0234 \ 0 \\
&0.9996 \ 1.0856 \ 0.6534 \ 0.3572 \ 0.3338 \\
&0.7238 \ 1.0166 \ 0.9292 \ 0.6330 \ 0.6096 \\
&0.4924 \ 0.7852 \ 1.1606 \ 0.8644 \ 0.8410 \\
&0.2124 \ 0.5052 \ 0.9374 \ 1.1444 \ 1.1210 \\
&0.2124 \ 0.5052 \ 0.9374 \ 1.1444 \ 1.1210
\end{align*}
\]

\[
\text{Columns 6 through 10}
\begin{align*}
&0.9996 \ 0.7238 \ 0.4924 \ 0.2124 \ 0.2124 \\
&1.0856 \ 1.0166 \ 0.7852 \ 0.5052 \ 0.5052 \\
&0.6534 \ 0.9292 \ 1.1606 \ 0.9374 \ 0.9374 \\
&0.3572 \ 0.6330 \ 0.8644 \ 1.1444 \ 1.1444 \\
&0.3338 \ 0.6096 \ 0.8410 \ 1.1210 \ 1.1210 \\
&0 \ 0.2758 \ 0.5072 \ 0.7872 \ 0.7872 \\
&0.2758 \ 0 \ 0.2314 \ 0.5114 \ 0.5114 \\
&0.5072 \ 0.2314 \ 0 \ 0.2800 \ 0.2800 \\
&0.7872 \ 0.5114 \ 0.2800 \ 0 \ 0.0000 \\
&0.7872 \ 0.5114 \ 0.2800 \ 0.0000 \ 0
\end{align*}
\]

\[
\text{vaf} = 0.7051
\]

\[
\text{addcon} = -0.8031
\]

As a variation on \texttt{cirfitac.m}, the m-file \texttt{cirfitac_ftarg.m} uses an additional fixed target matrix \texttt{TARG} to obtain the inflection points (and therefore,
TARG should provide a circular ordering. The syntax is otherwise the same as for cirfitac.m:

\[
[\text{fit}, \text{vaf}, \text{addcon}] = \text{cirfitac_ftarg}(\text{prox}, \text{inperm}, \text{targ})
\]

In the example below, an equally-(unit)-spaced circular ordering is used for TARG that is obtained from the utility function targcir.m; this strategy leads to a (slightly lower compared to cirfitac.m) VAF of .6670; here, the spacings around the circular structure between the placements for objects 1 and 2 is .3294; 2 and 3: .3204; 3 and 4: .2544; 4 and 5: .0344; 5 and 6: .2837; 6 and 7: .2084; 7 and 8: .3124; 8 and 9: .2701; 9 and 10: .0000; and back around between 10 and 1: .2109 (the circumference \( x_0 \) is 2.2241).

```matlab
>> load morse_digits.dat
>> targcircular = targcir(10)
targcircular =
     0     1     2     3     4     5     4     3     2     1
     1     0     1     2     3     4     5     4     3     2
     2     1     0     1     2     3     4     5     4     3
     3     2     1     0     1     2     3     4     5     4
     4     3     2     1     0     1     2     3     4     5
     5     4     3     2     1     0     1     2     3     4
     4     5     4     3     2     1     0     1     2     3
     3     4     5     4     3     2     1     0     1     2
     2     3     4     5     4     3     2     1     0     1
     1     2     3     4     5     4     3     2     1     0
```

```matlab
>> [fit, vaf, addcon] = cirfitac_ftarg(morse_digits, 1:10, targcircular)
```

fit =

Columns 1 through 7

\[
\begin{array}{cccccccc}
0 & 0.3294 & 0.6498 & 0.9043 & 0.9387 & 1.2224 & 0.7934 \\
0.3294 & 0 & 0.3204 & 0.5748 & 0.6903 & 0.8929 & 1.1014 \\
0.6498 & 0.3204 & 0 & 0.2544 & 0.2888 & 0.5725 & 0.7809 \\
0.9043 & 0.5748 & 0.2544 & 0 & 0.0344 & 0.3181 & 0.5265 \\
0.9387 & 0.6093 & 0.2888 & 0.0344 & 0 & 0.2837 & 0.4921 \\
1.2224 & 0.8929 & 0.5725 & 0.3181 & 0.2837 & 0 & 0.2084 \\
0.7934 & 1.1014 & 0.7809 & 0.5265 & 0.4921 & 0.2084 & 0 \\
0.4810 & 0.8104 & 1.0934 & 0.8389 & 0.8045 & 0.5208 & 0.3124 \\
0.2109 & 0.5403 & 0.8607 & 1.1091 & 1.0747 & 0.7910 & 0.5826 \\
0.2109 & 0.5403 & 0.8607 & 1.1151 & 1.0747 & 0.7910 & 0.5826 \\
\end{array}
\]

Columns 8 through 15
3.1. The Mechanics of CUS

Columns 8 through 10

<p>| | | | |</p>
<table>
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<tr>
<th></th>
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<tr>
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</table>

vaf =

0.6670

addcon =

-0.8317

The use of a fixed circular target matrix in cirfitac_ftarg.m (as opposed to finding one internally as is done in cirfit.m and cirfitac.m), could lead to small anomalies in the results, and the user should be thus prepared when using cirfitac_ftarg.m. In the example just given, for instance, the (4,10) value (of 1.1151) should probably be 1.1091 to match the (4,9) entry and the fact that 9 and 10 are at tied locations — however, the equally-spaced-circular-target distance from 10 to 4 is shorter clockwise (at a value of 4) than counter-clockwise (at a value of 6), and so the (4,10) value of 1.1151 is taken clockwise (as opposed to 1.1091 if taken counter-clockwise).

The function unicirac.m

The function m-file, unicirac.m, carries out a circular unidimensional scaling of a symmetric dissimilarity matrix (with the estimation of an additive constant) using an iterative quadratic assignment strategy (and thus, is an analogue of uniscalqa.m for the LUS task). We begin with an equally-spaced circular target constructed using the m-file targcir.m (that could be invoked with the command targcir(10)), a (random) starting permutation, and then use a sequential combination of the pairwise interchange/rotation/insertion heuristics; the target matrix is re-estimated based on the identified (locally optimal) permutation. The whole process is repeated until no changes can be made in the target or the identified (locally optimal) permutation. The explicit usage syntax is

[fit, vaf, outperm, addcon] = unicirac(prox, inperm, kblock)
where the various terms should now be familiar. INPERM is a given starting permutation (assumed to be around the circle) of the first \( n \) integers; FIT is the least-squares optimal matrix (with variance-accounted-for of VAF) to PROX having the appropriate circular form for the row and column object ordering given by the final permutation OUTPERM. The spacings between the objects are given by the diagonal entries in FIT (and the extreme \((1,n)\) entry in FIT). KBLOCK defines the block size in the use the iterative quadratic assignment routine. The additive constant for the model is given by ADDCON.

The problem of local optima is much more severe in CUS than in LUS. The heuristic identification of inflection points and the relevant spacings can vary slightly depending on the "equivalent" orderings identified around a circular structure. The example given below was identified as the best achievable (and for some multiple number of times) over 100 random starting permutations for INPERM; with its VAF of 71.90%, it is apparently the best "attainable". Given the (equivalent to the) identity permutation identified for outperm, the substantive interpretation for this representation is fairly clear — we have a nicely interpretable ordering of the Morse code symbols around a circular structure involving a regular replacement of dashes by dots moving clockwise until the symbol containing all dots is reached, and then a subsequent replacement of the dots by dashes until the initial symbol containing all dashes is reached.

```matlab
>> [fit,vaf,outperm,addcon] = unicirac(morse_digits,randperm(10),2)
```

```
fit =

Columns 1 through 6

0 0.0247 0.3620 0.6413 0.9605 1.1581
0.0247 0 0.3373 0.6165 0.9358 1.1334
0.3620 0.3373 0 0.2793 0.5985 0.7961
0.6413 0.6165 0.2793 0 0.3193 0.5169
0.9605 0.9358 0.5985 0.3193 0 0.1976
1.1581 1.1334 0.7961 0.5169 0.1976 0
1.1581 1.1334 0.7961 0.5169 0.1976 0.0000
1.0358 1.0606 1.0148 0.7355 0.4163 0.2187
0.7396 0.7643 1.1016 1.0318 0.7125 0.5149
0.3883 0.4131 0.7503 1.0296 1.0638 0.8662

Columns 7 through 10

1.1581 1.0358 0.7396 0.3883
1.1334 1.0606 0.7643 0.4131
0.7961 1.0148 1.1016 0.7503
0.5169 0.7355 1.0318 1.0296
0.1976 0.4163 0.7125 1.0638
0.0000 0.2187 0.5149 0.8662
```
3.2. Circular Multidimensional Scaling

The discussion in previous sections has been restricted to the fitting of a single circular unidimensional structure to a symmetric proximity matrix. Given the type of computational approach developed for carrying out this task (and, in particular, because of its lack of dependence on the presence of non-negative proximities), extensions are very direct to the use of multiple unidimensional structures through a process of successive residualization of the original proximity matrix. The fitting of two CUS structures to a proximity matrix generalizes (3.1) to the form

\[ \sum_{i<j} (p_{ij} - \min\{|x_j - x_i|, x_{0j} - |x_j - x_i|\} - c_1 - \min\{|x_j - x_i|, x_{0j} - |x_j - x_i|\} - c_2)^2. \]

The attempt to minimize (3.3) could proceed with the fitting of a single CUS structure to \( \{p_{ij}\} \), \( \min\{|x_j - x_i|, x_{0j} - |x_j - x_i|\} - c_1 \), using the computational strategy of Section 3.1, and once obtained, fitting a second CUS structure, \( \min\{|x_j - x_i|, x_{0j} - |x_j - x_i|\} - c_2 \) to the residual matrix, \( p_{ij} - \min\{|x_j - x_i|, x_{0j} - |x_j - x_i|\} - c_1 \). The process would then cycle by repetitively fitting the residuals from the second circular structure by the first, and the residuals from the first circular structure by the second, until the sequence converges. In any event, obvious extensions exist for (3.3) to the inclusion of more than two CUS structures, or to some mixture of, say, LUS and CUS forms in the spirit of Carroll and Pruzansky’s (1975, 1980) hybrid models.

The m-function, \texttt{bicirac.m}, is a two-(or bi-)dimensional scaling strategy for the \( L_2 \) loss function of (3.3), and relies heavily on the m-function \texttt{unicirac.m} to fit each separate circular structure, including its associated additive constant. The
syntax is

```matlab
[find, vaf, targone, targtwo, outpermone, outpermtwo, addconone, ...
addcontwo] = bicirac(prox, inperm, kblock)
```

where most of the terms should be familiar from previous usage in, say, `biscalqa.m`

Again, `PROX` is the input proximity matrix (\(n \times n\) with a zero main diagonal and a
dissimilarity interpretation); `INPERM` is a given starting permutation of the first \(n\)
integers; `FIND` is the least-squares optimal matrix (with variance-accounted-for of
`VAF`) to `PROX` and is the sum of the two circular (anti-Robinson) matrices `TARGONE`
and `TARGTWO` based on the two row and column object orderings given by the final
permutations `OUTPERMONE` and `OUTPERMTWO`. `KBLOCK` defines the block size in the
use of the iterative quadratic assignment routine and `ADDCONONE` and `ADDCONTWO`
are the two additive constants for the two model components.

As an illustration of the results obtainable from the process just described,
using the Morse code data, the MATLAB output below gives the best (according
to a VAF of 92.18%) two-CUS representation obtained from 100 random starting
permutations for each of the circular components. The two CUS structures have
rather clear substantive interpretations: as with our example using `unicirac.m`
the first shows the regular replacement of dots by dashes moving around the closed
continuum; the second provides a perfect ordering around the closed continuum
according to ratios of dots to dashes or of dashes to dots and where adjacent pairs
of stimuli have dashes and dots exchanged one-for-one; i.e., for the adjacent stimuli
pairs moving clockwise, we have:

0:5 for \{−−−−−;•••••\} (0,5); 1:4 for \{•−−−−;−••••\} (1,6); 2:3
for \{•−−−−;−••••\} (2,7); 3:2 for \{••−−−;−−•••\} (3,8); and 1:4 for
\{−−−−;••••−\} (9,4).

The two additive constants \(c_1\) and \(c_2\) in (3.3) have values of -.7002 and .3521,
respectively. (As mentioned, the output given below represents the best two-CUS
structures obtained for 100 random starting permutations, but as might be expected
given the earlier computational results, the same type of local optima were observed
here as found in the fitting of a single CUS structure, i.e., several local optima
were generated from small differences in the estimation of inflection points and the
adjacent object spacings but with the identical object orderings around the closed
continua).

```
>> [find, vaf, targone, targtwo, outpermone, outpermtwo, addconone, ...
addcontwo] = bicirac(morse_digits, randperm(10), 2)
```

```
find =

Columns 1 through 6

0 0.9765 1.4869 1.9626 1.7586 1.7461
0.9765 0 0.8585 1.5836 1.7815 1.7340
```


### 3.2. Circular Multidimensional Scaling

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Columns 7 through 10

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vaf = 0.9218
targone =

Columns 1 through 6

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Columns 7 through 10

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Chapter 3. Circular Scaling

```
1.0933  1.1958  1.1396  0.8302
0.7905  0.8931  1.3187  1.1329
0.3890  0.4916  0.9172  1.2266
0  0.1026  0.5282  0.8376
0.1026  0  0.4256  0.7350
0.5282  0.4256  0  0.3094
0.8376  0.7350  0.3094  0

targtwo =

Columns 1 through 6

0  0.0491  0.1825  0.3852  0.5267  0.6148
0.0491  0  0.1334  0.3361  0.4776  0.5657
0.1825  0.1334  0  0.2027  0.3442  0.4324
0.3852  0.3361  0.2027  0  0.1415  0.2296
0.5267  0.4776  0.3442  0.1415  0  0.0882
0.6148  0.5657  0.4324  0.2296  0.0882  0
0.6001  0.6427  0.5094  0.3066  0.1652  0.0770
0.3855  0.4346  0.5679  0.5212  0.3798  0.2916
0.1270  0.1761  0.3095  0.5122  0.6382  0.5600
0.0598  0.1089  0.2423  0.4450  0.5865  0.6173

Columns 7 through 10

0.6001  0.3855  0.1270  0.0598
0.6427  0.4346  0.1761  0.1089
0.5094  0.5679  0.3095  0.2423
0.3066  0.5212  0.5122  0.4450
0.1652  0.3798  0.6382  0.5665
0.0770  0.2916  0.5500  0.6173
0  0.2146  0.4731  0.5403
0.2146  0  0.2585  0.3257
0.4731  0.2585  0  0.0672
0.5403  0.3257  0.0672  0

outpermone =

8  9  10  1  2  3  4  5  6  7

outpermtwo =

7  3  8  4  9  10  5  1  6  2

addconone =
```
3.2. Circular Multidimensional Scaling

-0.7002

\text{addcontwo} =

0.3521
Chapter 4

LUS for Two-Mode Proximity Data

The proximity data considered thus far for obtaining some type of structural representation have been assumed to be on one intact set of objects, \( S = \{O_1, \ldots, O_n\} \), and complete in the sense that proximity values are present between all object pairs. Suppose now that the available proximity data are two-mode, that is, between two distinct object sets, \( S_A = \{O_{1A}, \ldots, O_{n_A}\} \) and \( S_B = \{O_{1B}, \ldots, O_{n_B}\} \), containing \( n_A \) and \( n_B \) objects, respectively, and defined through an \( n_A \times n_B \) proximity matrix \( Q = \{q_{rs}\} \), where again, for convenience, we assume that the entries in \( Q \) are keyed as dissimilarities. We may wish to seek a joint structural representation of the set \( S_A \cup S_B \) (considered as a single object set \( S \) containing \( n_A + n_B = n \) objects), but one that is based only on the available proximities between the sets \( S_A \) and \( S_B \).

A two-mode (dissimilarity matrix) data set for illustrative purposes

To provide a specific example that will be used throughout this chapter as an illustration, Table 4.1 presents an \( 11 \times 9 \) two-mode proximity matrix \( Q \) on the absorption of light at 9 different wavelengths by 11 different cones (receptors) in goldfish retina (but in a row and column reordered form that will reflect the discussion to follow). These data are from Schiffman and Falkenberg (1968) (and reanalyzed by Schiffman, Reynolds, and Young, 1981, pp. 328–329), and were originally based on an unpublished doctoral dissertation by Marks (1965). The proximities in the table are (200 minus) the heights of ordinates for particular spectral frequencies as labeled by the columns, and thus, can be considered dissimilarities reflecting the closeness of a particular receptor to a particular wavelength. Using the original labeling of the rows as given in Schiffman and Falkenberg, the row permutation in Table 4.1 is \( (3,8,9,2,6,4,1,7,5,11,10) \); the column permutation is \( (4,9,6,5,1,7,2,8,3) \). The latter column permutation in its given order corresponds to wavelengths of (458):blue-indigo, (430):violet, (485):blue, (498):blue-green, (530):green, (540):green, (585):yellow, (610):orange, (660):red.
Table 4.1. The goldfish receptor.dat data file constructed from Schiffman and Falkenberg (1968)

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</table>

4.1 Reordering Two-Mode Proximity Matrices

Given an \( n_a \times n_b \) two-mode proximity matrix, \( Q \), defined between the two distinct sets, \( S_A \) and \( S_B \), it may be desirable separately to reorder the rows and columns of \( Q \) to display some type of pattern that may be present in its entries, or to obtain some joint permutation of the \( n (= n_a + n_b) \) row and column objects to effect some further type of simplified representation. These kinds of reordering tasks will be approached with a variant of the quadratic assignment heuristics of the LUS discussion in Chapter 1 applied to a square, \( (n_a + n_b) \times (n_a + n_b) \), proximity matrix, \( P^{(tm)} \), in which a two-mode matrix \( Q_{(dev)} \) and its transpose (where \( Q_{(dev)} \) is constructed from \( Q \) by deviating its entries from the mean proximity), form the upper-right- and lower-left-hand portions, respectively, with zeros placed elsewhere. (This use of zero in the presence of deviated proximities, appears a reasonable choice generally in identifying good reorderings of \( P^{(tm)} \). Without this type of deviation strategy, there would typically be no “mixing” of the row and column objects in the permutations that we would identify for the combined [row and column] object set.) Thus, for \( 0 \) denoting (an appropriately dimensioned) matrix of all zeros,

\[
P^{(tm)} = \begin{bmatrix} 0_{n_a \times n_a} & Q_{(dev)_{n_a \times n_b}} \\ Q'_{(dev)_{n_b \times n_a}} & 0_{n_b \times n_b} \end{bmatrix},
\]

is the (square) \( n \times n \) proximity matrix subjected to a simultaneous row and column reordering, which in turn will induce separate row and column reorderings for the original two-mode proximity matrix \( Q \).

The m-file, `ordertm.m`, implements a quadratic assignment reordering heuristic on the derived matrix \( P^{(tm)} \), with usage

\begin{verbatim}
[outperm,rawindex,allperms,index,squareprox] = ... ordertm(proxtm,targ,inperm,kblock)
\end{verbatim}

where the two-mode proximity matrix `PROXTM` (with its entries deviated from the
4.2 Fitting a Two-Mode Unidimensional Scale

mean proximity within the use of the m-file) forms the upper-right- and lower-left-hand portions of a defined square \((n \times n)\) proximity matrix \((SQUAREPROX)\) with a dissimilarity interpretation, and with zeros placed elsewhere \((n = \text{number of rows} + \text{number of columns of PROXTM} = n_a + n_b)\); three separate local operations are used to permute the rows and columns of the square proximity matrix to maximize the cross-product index with respect to a square target matrix \(TARG\): (a) pairwise interchanges of objects in the permutation defining the row and column order of the square proximity matrix; (b) the insertion of from 1 to \(K\)-BLOCK \((\text{which is less than or equal to } n - 1)\) consecutive objects in the permutation defining the row and column order of the data matrix; (c) the rotation of from 2 to \(K\)-BLOCK \((\text{which is less than or equal to } n - 1)\) consecutive objects in the permutation defining the row and column order of the data matrix. INPERM is the beginning input permutation \((\text{a permutation of the first } n \text{ integers}); PROXTM is the two-mode \((n_a \times n_b)\) input proximity matrix; TARG is the \((n \times n)\) input target matrix. OUTPERM is the final permutation of SQUAREPROX with the cross-product index RAWINDEX with respect to \(TARG\). ALLPERMS is a cell array containing INDEX entries corresponding to all the permutations identified in the optimization from ALLPERMS\{1\} = INPERM to ALLPERMS\{INDEX\} = OUTPERM.

In the example to follow, ordertm.m, is used on the dissimilarity matrix of Table 4.1. The square equally-spaced target matrix is obtained from the LUS utility, targlin.m. A listing of the \((\text{reordered})\) matrix, squareprox(outperm,outperm), if given, would show clearly the unidimensional pattern for a two-mode data matrix that will be explicitly fitted in the next section of this chapter.

```
>> load goldfish_receptor.dat
>> [outperm,rawindex,allperms,index,squareprox] = ...
   ordertm(goldfish_receptor,targlin(20),randperm(20),2);
>> outperm

outperm =

Columns 1 through 10
20 11 10 19 9 18 8 7 17 16

Columns 11 through 20
6 5 4 15 14 13 3 12 2 1
```

4.2 Fitting a Two-Mode Unidimensional Scale

It is possible to fit through iterative projection, best-fitting \((\text{in the } L_2\text{-norm})\) unidimensional scales to two-mode proximity data based on a given permutation of the combined row and column object set. Specifically, if \(\rho(\cdot)\) denotes some given permutation of the first \(n\) integers \((\text{where the first } n_a \text{ integers denote row objects labeled } 1, 2, \ldots, n_a, \text{ and the remaining } n_b \text{ integers denote column objects, labeled } n_a + 1, n_a + 2, \ldots, n_a + n_b(= n))\), we seek a set of coordinates, \(x_1 \leq x_2 \leq \cdots \leq x_n\),
such that using the reordered square proximity matrix, \( P_{\rho_0}^{(tm)} = \{ p_{\rho_0(i),\rho_0(j)}^{(tm)} \} \), the least-squares criterion

\[
\sum_{i,j=1}^{n} w_{\rho_0(i),\rho_0(j)}(p_{\rho_0(i),\rho_0(j)}^{(tm)} - |x_j - x_i|)^2,
\]

is minimized, where \( w_{\rho_0(i),\rho_0(j)} = 0 \) if \( \rho_0(i) \) and \( \rho_0(j) \) are both row or both column objects, and = 1 otherwise. The entries in the matrix fitted to \( P_{\rho_0}^{(tm)} \) are based on the absolute coordinate differences (and which correspond to nonzero values of the weight function \( w_{\rho_0(i),\rho_0(j)} \)), and thus satisfy certain linear inequality constraints generated from how the row and column objects are intermixed by the given permutation \( \rho_0(\cdot) \). To give a schematic representation of how these constraints are generated, suppose \( r_1 \) and \( r_2 \) (\( c_1 \) and \( c_2 \)) denote two arbitrary row (column) objects, and suppose the following \( 2 \times 2 \) matrix represents what is to be fitted to the four proximity values present between \( r_1, r_2 \) and \( c_1, c_2 \):

\[
\begin{array}{c|cc}
 & c_1 & c_2 \\
r_1 & a & b \\
r_2 & c & d \\
\end{array}
\]

Depending on how these four objects are ordered (and intermixed) by the permutation \( \rho_0(\cdot) \), certain constraints must be satisfied by the entries \( a, b, c, \) and \( d \). The representative constraints are given schematically below according to the types of intermixing that might be present:

(a) \( r_1 < r_2 < c_1 < c_2 \) implies \( a + d = b + c \);
(b) \( r_1 < c_1 < r_2 < c_2 \) implies \( a + c + d = b \);
(c) \( r_1 < c_1 < c_2 < r_2 \) implies \( a + c = b + d \);
(d) \( r_1 < r_2 < c_1 \) implies \( c \leq a \);
(e) \( r_1 < c_1 < c_2 \) implies \( a \leq b \).

The confirmatory unidimensional scaling of a two-mode proximity matrix (based on iterative projection using a given permutation of the row and column objects) is carried out with the m-file, \texttt{linfittm}, with usage

\[
[\text{fit}, \text{diff}, \text{rowperm}, \text{colperm}, \text{coord}] = \text{linfittm}(\text{proxtm}, \text{inperm})
\]

Here, \( \text{PROXTM} \) is the two-mode proximity matrix, and \( \text{INPERM} \) is the given ordering of the row and column objects pooled together; \( \text{FIT} \) is an \( n_a \times n_b \) matrix of absolute coordinate differences fitted to \( \text{PROXTM} \), with \( \text{DIFF} \) being the (least-squares criterion) sum of squared discrepancies between \( \text{FIT} \) and \( \text{PROXTM} \), \( \text{ROWPERM} \) and \( \text{COLPERM} \) are the row and column object orderings derived from \( \text{INPERM} \). The \( (n_a + n_b) = n \) coordinates (ordered with the smallest such coordinate set at a value of zero) are given in \( \text{COORD} \).

The example given below uses a permutation obtained from \texttt{ordertm.m} on the data matrix \texttt{goldfish_receptor.dat}.

\[
>> [\text{fit}, \text{diff}, \text{rowperm}, \text{colperm}, \text{coord}] = \ldots
\]
### 4.2. Fitting a Two-Mode Unidimensional Scale

```
linfittn(goldfish_receptor,outperm);
>> fit

fit =
```

- **Columns 1 through 6**
  - [Values listed here]
  - [Values listed here]
  - [Values listed here]
  - [Values listed here]

- **Columns 7 through 9**
  - [Values listed here]
  - [Values listed here]
  - [Values listed here]

```
>> diff
diff =
    1.4372e+005
```

```
>> rowperm'
ans =
```

- **Columns 1 through 10**
  - [Values listed here]
  - [Values listed here]
  - [Values listed here]
Column 11

1

>> colperm'
ans =
     9 8 7 6 5 4 3 2 1

>> coord'
ans =

Columns 1 through 6
     0 27.7467 38.8578 47.3636 64.4133 77.6290

Columns 7 through 12
    82.0890 84.6355 132.9091 141.5455 151.4133 156.0800

Columns 13 through 18
    172.9689 202.1818 217.2727 240.1818 259.6356 259.6363

Columns 19 through 20
    286.9689 295.1911

In complete analogy with the LUS discussion (where the m-file linfitac.m generalizes linfit.m by fitting an additive constant along with the absolute coordinate differences), the more general unidimensional scaling model can be fitted with an additive constant using the m-file, linfittmac.m. Specifically, we now seek a set of coordinates, \(x_1 \leq x_2 \leq \cdots \leq x_n\), and an additive constant \(c\), such that using the reordered square proximity matrix, \(P^{(tm)}\), the least-squares criterion

\[
\sum_{i,j=1}^{n} w_{\rho_0(i),\rho_0(j)}(p^{(tm)}_{\rho_0(i),\rho_0(j)} + c - |x_j - x_i|)^2,
\]

is minimized, where again \(w_{\rho_0(i),\rho_0(j)} = 0\) if \(\rho_0(i)\) and \(\rho_0(j)\) are both row or both column objects, and = 1 otherwise. The m-file usage is

\[\text{[fit,vaf,rowperm,colperm,addcon,coord]} = \text{linfittmac(proxtm,inperm)}\]

and does a confirmatory two-mode fitting of a given unidimensional ordering of the
4.2. Fitting a Two-Mode Unidimensional Scale

Row and column objects of a two-mode proximity matrix PROXTM using the Dykstra-Kaczmarz iterative projection least-squares method. The m-file linfittmac.m differs from linfittm.m by including the estimation of an additive constant, and thus allowing VAF to be legitimately given as the goodness-of-fit index (as opposed to just DIFF as we did in linfittm.m). Again, INPERM is the given ordering of the row and column objects together; FIT is an \( n_a \) (number of rows) by \( n_b \) (number of columns) matrix of absolute coordinate differences fitted to PROXTM(ROWPERM,COLPERM); ROWPERM and COLPERM are the row and column object orderings derived from INPERM. The estimated additive constant ADDCON can be interpreted as being added to PROXTM (or alternatively subtracted from the fitted matrix FIT).

The same exemplar permutation is used below (as was for linfittm.m); following the MATLAB output that now includes the additive constant of -55.0512 and the VAF of .8072, the two unidimensional scalings (in their coordinate forms) are provided in tabular form with an explicit indication of what is a row object (R) and what is a column object (C).

```matlab
>> [fit,vaf,rowperm,colperm,addcon,coord] = ... linfittmac(goldfish_receptor,outperm);
>> vaf

vaf =

0.8072

>> rowperm'

ans =

Columns 1 through 10

11  10   9   8   7   6   5   4   3   2

Column 11

1

>> colperm'

ans =

9   8   7   6   5   4   3   2   1

>> addcon

addcon =
```
-55.0512

>> coord'

ans =

Columns 1 through 6
   0  16.7584  27.1305  27.9496  41.1914  46.4762

Columns 7 through 12
   47.9363  49.2521  82.8626  91.1532  91.9133  96.1573

Columns 13 through 18
 113.0462 122.1074 137.1983 160.1074 166.6057 166.6124

Columns 19 through 20
 178.1118 186.3341

4.3 Multiple LUS Reorderings and Fittings

Two m-files are provided that put together the (quadratic assignment) reordering of a two-mode rectangular proximity matrix with the fitting of the unidimensional scale(s). The first, uniscaltmac.m, combines the use of ordertm.m and linfittmac.m along with (re)estimations of the (originally equally-spaced) target matrix using the coordinates obtained until the identified permutation stabilizes. The usage includes the same terms as for the encompassing m-files:

[fit, vaf, outperm, rowperm, colperm, addcon, coord] = ...
uniscaltmac(proxtm, inperm, kblock)

The second m-file, biscalmac.m, finds and fits, through successive residualization, the sum of two linear unidimensional scales using iterative projection to a two-mode proximity matrix in the $L_2$-norm based on permutations identified through the use of iterative quadratic assignment. The usage has the form

[find, vaf, targone, targtwo, outpermone, outpermtwo, ...
   rowpermone, colpermone, rowpermtwo, colpermtwo, addconone, ...
   addcontwo, coordone, coordtwo, axes] = ...
biscalmac(proxtm, inpermone, inpermtwo, kblock, nopt)

Most of the terms should be obvious from earlier usage statements: the $n \times 2$ matrix, AXES, gives the two-dimensional plotting coordinates for the combined row and column object set. As was allowed in the bidimensional scaling routine biscalqa.m,
4.3. Multiple LUS Reorderings and Fittings

Table 4.2. The two unidimensional scalings of the goldfish_receptor data

<table>
<thead>
<tr>
<th>color</th>
<th>number</th>
<th>R or C</th>
<th>no constant</th>
<th>with constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>red (660)</td>
<td>20</td>
<td>C</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>R</td>
<td>27.7467</td>
<td>16.7584</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>R</td>
<td>38.8578</td>
<td>27.1305</td>
</tr>
<tr>
<td>orange (610)</td>
<td>19</td>
<td>C</td>
<td>47.3636</td>
<td>27.9496</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>R</td>
<td>64.4133</td>
<td>41.1914</td>
</tr>
<tr>
<td>yellow (585)</td>
<td>18</td>
<td>C</td>
<td>77.6290</td>
<td>46.4762</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>R</td>
<td>82.0890</td>
<td>47.9363</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>R</td>
<td>84.6355</td>
<td>49.2521</td>
</tr>
<tr>
<td>green (540)</td>
<td>17</td>
<td>C</td>
<td>132.9091</td>
<td>82.8626</td>
</tr>
<tr>
<td>green (530)</td>
<td>16</td>
<td>C</td>
<td>141.5455</td>
<td>91.1532</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>R</td>
<td>151.4133</td>
<td>91.9133</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>R</td>
<td>156.0800</td>
<td>96.1573</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>R</td>
<td>172.9689</td>
<td>113.0462</td>
</tr>
<tr>
<td>blue-green (490)</td>
<td>15</td>
<td>C</td>
<td>202.1818</td>
<td>122.1074</td>
</tr>
<tr>
<td>blue (485)</td>
<td>14</td>
<td>C</td>
<td>217.2727</td>
<td>137.1983</td>
</tr>
<tr>
<td>violet (430)</td>
<td>13</td>
<td>C</td>
<td>240.1818</td>
<td>160.1074</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>R</td>
<td>259.6356</td>
<td>166.6057</td>
</tr>
<tr>
<td>blue-indigo (458)</td>
<td>12</td>
<td>C</td>
<td>259.6363</td>
<td>166.6124</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>R</td>
<td>286.9689</td>
<td>178.1118</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>R</td>
<td>295.1911</td>
<td>186.3341</td>
</tr>
</tbody>
</table>

the variable NOPT controls the confirmatory or exploratory fitting of the unidimensional scales; a value of NOPT = 0 will fit in a confirmatory mode the two scales indicated by INPERMONE and INPERMTWO; a value of NOPT = 1 uses iterative QA to locate the better permutations to fit.

An example of using biscaltmac.m follows, leading to a two-dimensional scaling of the goldfish_receptor data with a VAF of .9620. A two-dimensional graphical representation of the coordinates will be given in the next section after the necessary plotting utility, biplottm.m, is introduced.

```matlab
>> load goldfish_receptor.dat
>> [find,vaf,targone,targtwo,outpermone,outpermtwo,...
rowpermone,colpermone,rowpermtwo,colpermtwo,addconone,...
addcontwo,coordone,coordtwo,axes] = ...
biscaltmac(goldfish_receptor,randperm(20),randperm(20),2,1);
>> vaf

vaf =

0.9620
```
>> outpermone

outpermone =

Columns 1 through 10

  20  11  10  19   9  18   8  7  17  16

Columns 11 through 20

   6   5   4  15  14  13   3  12   2   1

>> coordone'

ans =

Columns 1 through 6

   0  5.3813 29.6923 29.6923 47.3362 47.3362

Columns 7 through 12

  47.3362 47.3362 80.1506 88.7578 91.1164 100.5008

Columns 13 through 18

115.5844 131.4868 141.7676 149.8850 160.3825 160.3825

Columns 19 through 20

173.7454 181.0428

>> outpermtwo

outpermtwo =

Columns 1 through 10

   3  20   1   2  13  10   9  19  12   8

Columns 11 through 20

  14  11  18   6  15   4  16  17   7

>> coordtwo'
4.3. Multiple LUS Reorderings and Fittings

ans =

Columns 1 through 6
    0     6.7276    6.7277    7.8975   14.0132   14.0891

Columns 7 through 12
   14.0891   27.5247   27.5247   30.1025   40.4679   40.4710

Columns 13 through 18
   49.4002   58.0796   58.0796   66.8364   72.2495   72.7100

Columns 19 through 20
   72.8142    90.6794

>> axes

axes =

    181.0428    6.7277
    173.7454    7.8975
    160.3825     0
    115.5844   66.8364
    100.5008   72.2495
     91.1164   58.0796
     47.3362   90.6794
     47.3362   30.1025
     47.3362   14.0891
     29.6923   14.0891
      5.3813   40.4710
    160.3825   27.5247
    149.8850   14.0132
    141.7676   40.4679
    131.4868   58.0796
     88.7578   72.7100
     80.1506   72.8142
     47.3362   49.4002
     29.6923   27.5247
      0     6.7276
4.4 Some Useful Two-Mode Utilities

This section gives several miscellaneous m-functions that carry out various operations on a two-mode proximity matrix, and for which no other section of this monograph seemed appropriate. The first two, proxstdtm.m and proxrandtm.m, are very simple and provide standardized and randomly (entry-)permuted two-mode proximity matrices, respectively, that might be useful, for example, in testing the various m-functions we give. The syntax

\[
\text{[stanproxtm,stanproxmulttm] = proxstdtm(proxtm,mean)}
\]

is intended to suggest that STANPROXTM provides a linear transformation of the entries in PROXTM to a standard deviation of one and a mean of MEAN; STANPROXMULTTM is a multiplicative transformation so that the entries in this \(n_a \times n_b\) matrix have a sum-of-squares of \(n_a n_b\). For the second utility m-function

\[
\text{[randproxtm] = proxrandtm(proxtm)}
\]

implies that the two-mode matrix RANDPROXTM has its entries as a random permutation of the entries in PROXTM.

A third utility function, proxmontm.m, provides a monotonically transformed two-mode proximity matrix that is close in a least-squares sense to a given input two-mode matrix. The syntax is

\[
\text{[monproxpermuttm, vaf, diff] = proxmontm(proxpermuttm,fittedtm)}
\]

Here, PROXPERMUTTM is the original input two-mode proximity matrix (which may have been subjected to initial row and column permutations, hence the suffix ‘PERMUTTM’), and FITTEDTM is a given two-mode target matrix; the output matrix MONPROXPERMUTTM is closest to FITTEDTM in a least-squares sense and obeys the order constraints obtained from each pair of entries in PROXPERMUTTM (and where the inequality constrained optimization is carried out using the Dykstra-Kaczmarz iterative projection strategy); as usual, VAF indicates how much variance in MONPROXPERMUTTM can be accounted for by FITTEDTM; finally, DIFF is the value of the least-squares loss function and is the sum of squared differences between the entries in MONPROXPERMUTTM and FITTEDTM. We will give an application of an m-file incorporating proxmonton.m when we suggest in the next Section 4.5, a way of implementing two-dimensional, two-mode nonmetric multidimensional scaling.

A final utility function, biplottm.m, plots the combined row and column object set using the coordinates given in, for example, the \(n \times 2\) output matrix AXES as output from the m-file of the last section, biscaltmac.m. The usage syntax is

\[
\text{biplottm(axes,nrow,ncol)}
\]

Here, the number of rows (columns) is NROW (NCOL), and \(n\) is the sum of NROW and NCOL. The first NROW rows of the \(n \times 2\) matrix AXES give the row object coordinates; the last NCOL rows of AXES give the column object coordinates. The plotting symbol for rows is a circle (o); for columns it is an asterisk (*). The labels for rows are from 1 to NROW; those for columns are from 1 to NCOL.
4.5 Two-mode Nonmetric Bidimensional Scaling

By uniting the utility function proxmon.m with biscaltmac.m, we can construct an m-file, bimonscaltmac.m, that carries out a nonmetric bidimensional scaling of a two-mode proximity matrix in the city-block metric. The usage is the same.
as that of `biscaltmac.m` in Section 4.3, except for the additional output matrix `MONPROXTM` that is a monotonic transformation of the original two-mode proximity matrix `PROXTM`:

```
[...], monproxtm] = bimonscaltmac(...)
```

We give an example below using the same `goldfish_receptor.dat` matrix; the VAF has increased (slightly) to .9772. The joint plot of the row and column object set is given in Figure 4.2, and closely resembles Figure 4.1 obtained without the use of a monotonic transformation.

```
>> [find, vaf, targone, targtwo, outpermone, outpermtwo,...
rowpermone, colpermone, rowpermtwo, colpermtwo, addconone,...
addcontwo, coordone, coordtwo, axes, monproxtm] = ...
bimonscaltmac(goldfish_receptor,1:20,1:20,1,1);

>> vaf

vaf =

0.9772

>> outpermone

outpermone =

Columns 1 through 12
   1   2  12   3  13  14   5   6   16  17

Columns 13 through 20
     7  8  18   9  19  10  11  20

>> outpermtwo

outpermtwo =

Columns 1 through 12
   3  20   1   2  13  10  9  19  8  12  14  11

Columns 13 through 20
    18   6  15   4   5  16  17   7

>> coordone'
```
4.5. Two-mode Nonmetric Bidimensional Scaling

ans =

Columns 1 through 7

0  9.3971  24.9145  25.6090  33.0810  41.8175  52.5796

Columns 8 through 14

64.9000  81.8939  86.7370  91.3614  95.4814  128.0062  129.3501

Columns 15 through 20

129.3501  129.3501  144.3894  144.6789  166.2783  172.0484

>> coordtwo

ans =

Columns 1 through 7


Columns 8 through 14

25.9193  25.9193  25.9193  35.1740  37.0213  48.2031  53.3228

Columns 15 through 20

53.3228  62.5256  66.5361  67.7899  67.7899  83.2680

>> monproxtm

monproxtm =

Columns 1 through 7

58.7038  63.2885  108.3069  141.6262  189.0671  189.0671  203.9042
58.7038  63.2885  82.1362  108.3069  181.2299  189.0671  192.8095
58.7038  58.7038  85.7059  124.9383  167.7111  167.7111  181.2299
108.3069  108.3069  82.1362  63.2885  58.7038  72.2093  108.3069
124.9383  136.7608  120.4397  82.1362  58.7038  48.4205  91.1332
120.4397  141.6262  97.3034  72.2093  58.7038  58.7038  82.1362
189.0671  189.0671  154.9866  141.6262  141.6262  108.3069  113.1332
138.5167  154.9866  124.9383  138.5167  120.4397  108.3069  58.7038
141.6262  120.4397  141.6262  141.6262  120.4397  121.7048  67.2074
### Chapter 4. LUS for Two-Mode Proximity Data

| 167.7111 | 141.6262 | 167.7111 | 167.7111 | 141.6262 | 138.5167 | 72.2093 |
| 189.1641 | 193.2795 | 162.2199 | 166.2670 | 141.6262 | 138.5167 | 82.1362 |

Columns 8 through 9

| 196.9665 | 208.6756 |
| 189.0671 | 200.9956 |
| 181.2299 | 191.5023 |
| 154.9866 | 197.4823 |
| 141.6262 | 189.0671 |
| 124.9383 | 167.7111 |
| 108.3069 | 154.9866 |
| 48.4214  | 82.1362  |
| 58.7038  | 63.2885  |
| 63.2885  | 82.1362  |
| 63.2885  | 72.2093  |
Figure 4.2. Two-dimensional joint biplot for the goldfish receptor data obtained using bimonscalmac.m and biplottm.m
Chapter 4. LUS for Two-Mode Proximity Data
Part II

The Representation of Proximity Matrices by Tree Structures
Introduction to
Graph-Theoretic
Representational Structures

Various methods of data representation based on graph-theoretic structures have been developed over the last several decades for explaining the pattern of information potentially present in a single (or possibly, in a collection of) numerically given proximity matrix(ies), each defined between pairs of objects from a single set, or in some cases, between the objects from several distinct sets (for example, see Carroll, 1976; Carroll, Clark, & DeSarbo, 1984; Carroll & Pruzansky, 1980; De Soete, 1983, 1984a,b,c; De Soete, Carroll, & DeSarbo, 1987; De Soete, DeSarbo, Furnas, & Carroll, 1984; Hutchinson, 1989; Klauer & Carroll, 1989, 1991; Hubert & Arabie, 1995b). Typically, a specific class of graph-theoretic structures is assumed capable of representing the proximity information, and the proposed method seeks a member from the class producing a reconstructed set of proximities that are as close as possible to the original. The most prominent graph-theoretic structures used are those usually referred to as ultrametrics and additive trees, and these will be the primary emphasis here as well.

Although a variety of strategies have been proposed for locating good exemplars from whatever class of graph-theoretic structures is being considered, one approach has been to adopt a least squares criterion in which the class exemplar is identified by attempting to minimize the sum of squared discrepancies between the original proximities and their reconstructions obtained through the use of the particular structure selected by the data analyst. One common implementation of the least-squares optimization strategy has been defined by the usual least-squares criterion but augmented by some collection of penalty functions that seek to impose whatever constraints are mandated by the structural representation being sought. Then, through the use of some unconstrained optimization scheme (e.g., steepest descent, conjugate gradients), an attempt is made to find both (a) the particular constraints that should be imposed to define the specific structure from the class, and (b) the reconstructed proximities based on the structure finally identified. The resulting optimization strategy is heuristic in the sense that there is no guarantee of global optimality for the final structural representation identified even within the chosen graph-theoretic class, because the particular constraints defining the selected
structure were located by a possibly reasonable but not verifiably optimal search strategy that was (implicitly) implemented in the course of the process of optimization. A second implementation of the least-squares optimization approach, and the one that we will concentrate on exclusively, is based on the type of iterative projection strategy already illustrated in conjunction with linear unidimensional scaling (LUS) (see the addendum Section 1.4 on solving linear inequality constrained least-squares tasks), and developed in detail for the graph-theoretic context by Hubert and Arabie (1995b). In its non-heuristic form, iterative projection allows the reconstruction of a set of proximities based on a fixed collection of constraints implied by whatever specific graph-theoretic structure has been selected for their representation. As in LUS, successive (or iterative) projections onto closed convex sets are carried out that are defined by the collection of given constraints implied by the structural representation chosen. Thus, the need for penalty terms is avoided and there is no explicit use of gradients in the attendant optimization strategy; also, it is fairly straightforward to incorporate a variety of different types of constraints that may be auxiliary to those generated from the given structural representation but none-the-less of interest to impose on the reconstruction.

As a least squares optimization strategy (in a non-heuristic form), iterative projection assumes that whatever constraint set is to be applied is completely known prior to its application. However, just as the various penalty-function and gradient-optimization techniques have been turned into heuristic search strategies for the particular structures of interest by allowing the collection of constraints to vary over the course of the optimization process, we attempt the same in using iterative projection to find the better-fitting ultrametrics and additive trees for a given proximity matrix. Thus, in addition to carrying out a least squares task subject to given structural constraints, iterative projection will be considered as one possible heuristic search strategy (and an alternative to those heuristic methods that have been suggested in the literature and based exclusively on the use of some type of penalty function) for locating the actual constraints to impose, and therefore, to identify the specific form of the structural representation sought.

The various least squares optimization tasks entailing both the identification of the specific form of the structural representation to adopt and the subsequent least squares fitting itself generally fall into the class of NP-hard problems (e.g., for ultrametric and additive trees, see Day, 1987, 1996; Krivánek, 1986; Krivánek & Morávek, 1986); thus, the best we can hope for is a heuristic extension of the iterative projection strategy leading to good but not necessarily optimal final structural representations within the general class of representations desired. As is standard with a reliance on such heuristic optimization methods, the use of multiple starting points will hopefully determine a set of local optima characterizing the better solutions attainable for a given data set. The presence of local optima in the use of any heuristic and combinatorially based optimization strategy is unavoidable, given the NP-hardness of the basic optimization tasks of interest and the general inability of (partial) enumeration methods (when available) to be computationally feasible for use on even moderate-sized data sets. The number of and variation in the local optima observable for any specific situation will obviously depend on the given data, the structural representation sought, and the heuristic search strategy used.
But whenever present, local optima may actually be diagnostic for the structure(s) potentially appropriate for characterizing a particular data set. Thus, their identification may even be valuable in explaining the patterning of the data and/or in noting the difficulties with adopting a specific representational form to help discern underlying structure.
Chapter 5

Ultrametrics for Symmetric Proximity Data

The task of hierarchical clustering can be characterized as a specific data analysis problem: given a set of \( n \) objects, \( S = \{O_1, \ldots, O_n\} \), and an \( n \times n \) symmetric proximity matrix \( P = \{p_{ij}\} \) (non-negative and with a dissimilarity interpretation), find a sequence of partitions of \( S \), denoted as \( P_1, P_2, \ldots, P_n \), satisfying the following:

(a) \( P_1 \) is the (trivial) partition where all \( n \) objects from \( S \) are placed into \( n \) separate classes;
(b) \( P_n \) is the (also trivial) partition where a single subset contains all \( n \) objects;
(c) \( P_k \) is obtained from \( P_{k-1} \) by uniting some pair of classes present in \( P_{k-1} \);
(d) the minimum levels at which object pairs first appear together within the same class should reflect the proximities in \( P \). Or more formally, if we define \( U^0 = \{u^0_{ij}\} = \min\{k-1 \mid \text{objects } O_i \text{ and } O_j \text{ appear within the same class in } P_k\} \), then if the partition hierarchy is representing the given proximities well, the entries in \( U^0 \) and \( P \) should be, for example, similarly ordered. We discuss the properties of matrices such as \( U^0 \) in more detail below.

To give an example, we performed a complete-link hierarchical clustering (using SYSTAT) on the number.dat proximity matrix used extensively in Part I, and obtained the following partitions of the object indices from 1 to 10 (remembering that these correspond to the digits 0 to 9):

\[
\begin{align*}
P_1 & : \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\} \\
P_2 & : \{\{3,5\}, \{1\}, \{2\}, \{4\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\} \\
P_3 & : \{\{3,5\}, \{4,10\}, \{1\}, \{2\}, \{6\}, \{7\}, \{8\}, \{9\}\} \\
P_4 & : \{\{3,5\}, \{4,7,10\}, \{1\}, \{2\}, \{6\}, \{8\}, \{9\}\} \\
P_5 & : \{\{3,5,9\}, \{4,7,10\}, \{1\}, \{2\}, \{6\}, \{8\}\} \\
P_6 & : \{\{3,5,9\}, \{4,7,10\}, \{6,8\}, \{1\}, \{2\}\} \\
P_7 & : \{\{3,5,9\}, \{4,7,10\}, \{6,8\}, \{1,2\}\} \\
P_8 & : \{\{3,5,9\}, \{4,6,7,8,10\}, \{1,2\}\} \\
P_9 & : \{\{3,4,5,6,7,8,9,10\}, \{1,2\}\} \\
P_{10} & : \{\{1,2,3,4,5,6,7,8,9,10\}\}
\end{align*}
\]

The matrix \( U^0 \) was constructed and saved as a \( 10 \times 10 \) matrix in the file numcltarg.dat, which will be used later in an example:
is equivalent to the A concept routinely encountered in discussions of hierarchical clustering is that of an ultrametric, which can be characterized here as any non-negative symmetric dissimilarity matrix for the objects in $S$, denoted generically by $U = \{u_{ij}\}$, where $u_{ij} = 0$ if and only if $i = j$, and $u_{ij} \leq \max[u_{ik}, u_{jk}]$ for all $1 \leq i, j, k \leq n$ (this last inequality, called the three-point or three-object condition, statement that for any distinct triple of subscripts, $i$, $j$, and $k$, the largest two proximities among $u_{ij}$, $u_{ik}$, and $u_{jk}$ are equal and [therefore] not less than the third). Any ultrametric can be associated with the specific partition hierarchy it induces, having the form $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_T$, where $\mathcal{P}_1$ and $\mathcal{P}_T$ are now the two trivial partitions that respectively contain all objects in separate classes and all objects in the same class, and $\mathcal{P}_k$ is formed from $\mathcal{P}_{k-1}$ ($2 \leq k \leq T$) by (agglomeratively) uniting certain (and possibly more than two) subsets in $\mathcal{P}_{k-1}$. For those subsets merged in $\mathcal{P}_{k-1}$ to form $\mathcal{P}_k$, all between-subset ultrametric values must be equal, and no less than any other ultrametric value associated with an object pair within a class in $\mathcal{P}_{k-1}$. Thus, individual partitions in the hierarchy can be identified by merely increasing a threshold variable starting at zero, and observing that $\mathcal{P}_k$ for $1 \leq k \leq T$ is defined by a set of subsets in which all within-subset ultrametric values are less than or equal to some specific threshold value, and all ultrametric values between subsets are strictly greater. Conversely, any partition hierarchy of the form $\mathcal{P}_1, \ldots, \mathcal{P}_T$ can be identified with the equivalence class of all ultrametric matrices that induce it. We note that if only a single pair of subsets can be united in $\mathcal{P}_{k-1}$ to form $\mathcal{P}_k$, $2 \leq k \leq T$, then $T = n$, and we could then revert to the characterization of a full partition hierarchy $\mathcal{P}_1, \ldots, \mathcal{P}_n$ used earlier.

Given some fixed partition hierarchy $\mathcal{P}_1, \ldots, \mathcal{P}_T$, there are an infinite number of ultrametric matrices that induce it, but all can be generated by (restricted) monotonic functions of what might be called the basic ultrametric matrix $U^0$ defined earlier. Explicitly, any ultrametric in the equivalence class whose members induce the same fixed hierarchy, $\mathcal{P}_1, \ldots, \mathcal{P}_T$, can be obtained by a strictly increasing monotonic function of the entries in $U^0$, where the function maps zero to zero. Moreover, because $u^0_{ij}$ for $i \neq j$ can be only one of the integer values from 1 to $T - 1$, each ultrametric in the equivalence class that generates the fixed hierarchy may be defined by one of $T - 1$ distinct values. When these $T - 1$ values are ordered from the smallest to the largest, the $(k - 1)^{st}$ smallest value corresponds to the partition $\mathcal{P}_k$ in the partition hierarchy $\mathcal{P}_1, \ldots, \mathcal{P}_T$, and implicitly to all object pairs that appear together for the first time within a subset in $\mathcal{P}_k$. 

\[
\begin{array}{cccccccccccc}
0 & 6 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
6 & 0 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 0 & 8 & 1 & 8 & 8 & 8 & 4 & 8 & 4 & 8 \\
9 & 9 & 8 & 0 & 8 & 7 & 3 & 7 & 8 & 2 & 8 & 4 \\
9 & 9 & 1 & 8 & 0 & 8 & 8 & 8 & 4 & 8 & 4 & 8 \\
9 & 9 & 8 & 7 & 8 & 0 & 7 & 5 & 8 & 7 & 5 & 8 \\
9 & 9 & 8 & 3 & 8 & 7 & 0 & 7 & 8 & 3 & 8 & 7 \\
9 & 9 & 8 & 7 & 8 & 5 & 7 & 0 & 8 & 7 & 8 & 5 \\
9 & 9 & 4 & 8 & 4 & 8 & 8 & 8 & 0 & 8 & 9 & 8 \\
9 & 9 & 8 & 2 & 8 & 7 & 3 & 7 & 8 & 0 & 2 & 9
\end{array}
\]
5.1. Fitting a Given Ultrametric in the $L_2$ Norm

To provide an alternative interpretation, the basic ultrametric matrix can also be characterized as defining a collection of linear equality and inequality constraints that any ultrametric in a specific equivalence class must satisfy. Specifically, for each object triple there is (a) a specification of which ultrametric values among the three must be equal plus two additional inequality constraints so that the third is not greater; (b) an inequality or equality constraint for every pair of ultrametric values based on their order relationship in the basic ultrametric matrix; and (c) an equality constraint of zero for the main diagonal entries in $U$. In any case, given these fixed equality and inequality constraints, standard $L_p$ regression methods (such as those given in Späth, 1991), could be adapted to generate a best-fitting ultrametric, say $U^* = \{u^*_{ij}\}$, to the given proximity matrix $P = \{p_{ij}\}$. Concretely, we might find $U^*$ by minimizing

$$
\sum_{i<j} (p_{ij} - u_{ij})^2, \quad \sum_{i<j} |p_{ij} - u_{ij}|, \quad \text{or possibly, } \max_{i<j} |p_{ij} - u_{ij}|
$$

(As a convenience here and later, it is assumed that $p_{ij} > 0$ for all $i \neq j$, to avoid the technicality of possibly locating best-fitting ‘ultrametrics’ that could violate the condition that $u_{ij} = 0$ if and only if $i = j$.)

5.1 Fitting a Given Ultrametric in the $L_2$ Norm

The function, `ultrafit.m`, with usage

```matlab
[fit, vaf] = ultrafit(prox, targ)
```

generates (using iterative projection based on the linear (in)equality constraints obtained from the fixed ultrametric — see Section 1.4) the best-fitting ultrametric in the $L_2$-norm (FIT) within the same equivalence class as that of a given ultrametric matrix $TARG$. The matrix $PROX$ contains the symmetric input proximities and $VAF$ is the variance-accounted-for (defined, as usual, by normalizing the obtained $L_2$-norm loss value):

$$
VAF = 1 - \frac{\sum_{i<j}(p_{ij} - u^*_{ij})^2}{\sum_{i<j}(p_{ij} - \bar{p})^2},
$$

where $\bar{p}$ is the mean off-diagonal proximity in $P$, and $U^* = \{u^*_{ij}\}$ is the best-fitting ultrametric.

In the example below, the target matrix is `numcltarg` obtained from the complete-link hierarchical clustering of `number`; the VAF generated by these ultrametric constraints is .4781. Comparing the target matrix `numcltarg` and `fit`, the particular monotonic function, say $f(\cdot)$, of the entries in the basic ultrametric matrix that generates the fitted matrix are: $f(1) = .0590$, $f(2) = .2630$, $f(3) = .2980$, $f(4) = .3065$, $f(5) = .4000$, $f(6) = .4210$, $f(7) = .4808$, $f(8) = .5535$, $f(9) = .6761$.

```matlab
load number.dat
load numcltarg.dat
[fit, vaf] = ultrafit(number, numcltarg)
```
fit =

Columns 1 through 6

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<td>0.6761</td>
<td>0.5535</td>
<td>0.2980</td>
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<td>0.2980</td>
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Columns 7 through 10

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<td>0</td>
<td>0.5535</td>
<td></td>
</tr>
<tr>
<td>0.2980</td>
<td>0.4808</td>
<td>0.5535</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

vaf =

0.4781

5.2 Finding an Ultrametric in the $L_2$ Norm

The m-file, ultrafnd.m, implements a heuristic search strategy using iterative projection to locate a best-fitting ultrametric in the $L_2$-norm. The method used is from Hubert and Arabie (1995b); this latter source should be consulted for the explicit algorithmic details implemented in ultrafnd.m (as well as for many of the other m-files to be presented). The m-file usage has the form

```
[find,vaf] = ultrafnd(prox,inperm)
```

where FIND is the ultrametric identified having variance-accounted-for VAF. The matrix PROX contains the symmetric input proximities; INPERM is a permutation that defines an order in which the constraints are considered over all object triples.
5.2. Finding an Ultrametric in the $L_2$ Norm

In the example below, for instance, \texttt{INPERM} is simply set as the MATLAB built-in random permutation function \texttt{randperm(n)} (using the size \texttt{n = 10} explicitly for the \texttt{number} illustration). Thus, the search can be rerun with the same specification but now using many different random starting sequences. Two such searches are shown below leading to VAFs of .4941 and .4781 (the latter is the same as obtained from fitting the best ultrametric in Section 5.1 using \texttt{numcltarg} for a fixed set of constraints; the former provides a slightly different and better-fitting ultrametric).

\[
\texttt{[find,vaf] = ultrafnd(number,randperm(10))}
\]

\texttt{find} =

\begin{verbatim}
Columns 1 through 6

0 0.7300 0.7300 0.7300 0.7300 0.7300
0.7300 0 0.5835 0.5835 0.5835 0.5835
0.7300 0.5835 0 0.5535 0.0590 0.5535
0.7300 0.5835 0.5535 0 0.5535 0.4808
0.7300 0.5835 0.0590 0.5535 0 0.5535
0.7300 0.5835 0.5535 0.4808 0.5535 0.4808
0.7300 0.5835 0.5535 0.2980 0.5535 0.4808
0.7300 0.5835 0.3065 0.5535 0.3065 0.5535
0.7300 0.5835 0.5535 0.2630 0.5535 0.4808

Columns 7 through 10

0.7300 0.7300 0.7300 0.7300
0.5835 0.5835 0.5835 0.5835
0.5535 0.5535 0.3065 0.5535
0.2980 0.4808 0.5535 0.2630
0.5535 0.5535 0.3065 0.5535
0.4808 0.4000 0.5535 0.4808
0 0.4808 0.5535 0.2980
0.4808 0 0.5535 0.4808
0.5535 0.5535 0 0.5535
0.2980 0.4808 0.5535 0
\end{verbatim}

\texttt{vaf} =

\begin{verbatim}
0.4941
\end{verbatim}

\[
\texttt{[find,vaf] = ultrafnd(number,randperm(10))}
\]

\texttt{find} =
### 5.3 Graphically Representing an Ultrametric

Once an ultrametric matrix has been identified, there are two common ways in which the information within the matrix might be displayed. The first is to perform a simple reordering of the rows and columns of the given matrix to make apparent the sequence of partitions being induced by the ultrametric. The form desired is typically called anti-Robinson (see, for example, Hubert and Arabie, 1994 [or Part III of this current text], for a very complete discussion of using and fitting such matrix orderings). When a matrix is in anti-Robinson form, the entries within each row (and column) are non-decreasing moving away from the main diagonal in either direction. As the example given below will show, any ultrametric matrix can be put into such a form easily (but nonuniquely). The second strategy for representing an ultrametric relies on the graphical form of an inverted tree (or as
5.3. Graphically Representing an Ultrametric

it is typically called in the classification literature, a dendrogram), and where one can read the values of the ultrametric directly from the displayed structure. We give an example of such a tree below (and provide at the end of this Chapter the \LaTeX\ code [within the picture environment] to generate this particular graphical structure).

To give the illustration of reordering an ultrametric matrix to display its anti-Robinson form, the example found in Section 5.2 with a VAF of .4941 will be used, along with a short m-file, \texttt{ultraorder.m}. This function implements a simple mechanism of first generating a unidimensional equally-spaced target matrix from the utility m-file \texttt{targlin.m}, and then reorders heuristically the given ultrametric matrix against this given target with the quadratic assignment functions \texttt{pairwiseqa.m} and \texttt{insertqa.m} (the latter uses the maximum block size of $n-1$ for \texttt{kblock}). The explicit usage is

\begin{verbatim}
[orderprox,orderperm] = ultraorder(prox)
\end{verbatim}

where \texttt{PROX} is assumed to be an ultrametric matrix; \texttt{ORDERPERM} is a permutation used to display the anti-Robinson form in \texttt{ORDERPROX}, where

\begin{verbatim}
orderprox = prox(orderperm,orderperm).
\end{verbatim}

\begin{verbatim}
load number.dat
[find,vaf] = ultrafnd(number,randperm(10));
[orderprox,orderperm] = ultraorder(find)
orderprox =

Columns 1 through 6

\begin{verbatim}
0 0.7300 0.7300 0.7300 0.7300 0.7300
0.7300 0 0.3065 0.3065 0.5535 0.5535
0.7300 0.3065 0 0.0590 0.5535 0.5535
0.7300 0.5535 0.5535 0.5535 0 0.2630
0.7300 0.5535 0.5535 0.5535 0.2980 0.4808
0.7300 0.5535 0.5535 0.5535 0.4808 0.5835
0.7300 0.5835 0.5835 0.5835 0.5835 0.5835
\end{verbatim}

Columns 7 through 10

\begin{verbatim}
0.7300 0.7300 0.7300 0.7300
0.5535 0.5535 0.5535 0.5835
0.5535 0.5535 0.5535 0.5835
0.5535 0.5535 0.5535 0.5835
0.2980 0.4808 0.4808 0.5835
\end{verbatim}

Chapter 5. Ultrametrics for Symmetric Proximity Data

The reordered matrix using the row and column order of 0 ≺ 8 ≺ 2 ≺ 4 ≺ 9 ≺ 3 ≺ 6 ≺ 7 ≺ 5 ≺ 1 is given below; here the blocks of equal-valued entries are highlighted, indicating the partition hierarchy (also given below) induced by the ultrametric.

<table>
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<th>2</th>
<th>4</th>
<th>9</th>
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<th>6</th>
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For the partition hierarchy just given, the alternative structure of a dendrogram (or tree) for its representation is given in the figure that follows. The terminal "nodes" of this structure, indicated by open circles, correspond to the ten digits; the filled circles are internal "nodes" reflecting the level at which certain new classes
in a partition hierarchy are constructed. For instance, using the calibration given
on the long vertical line at the left, a new class consisting of the digits \{9,3,6,7,5\}
is formed at level .48 by uniting the two classes \{9,3,6\} and \{7,5\}. Thus, in the
ultrametric matrix given earlier, the values between the entries in these two classes
are all a constant .48.

The dendrogram just given can be modified (or at least in how it should
be interpreted) to motivate the representational form of an additive tree to be
introduced in Chapter 6: (a) the values in the calibration along the long vertical
axis need to be halved; (b) all horizontal lines are now to be understood as having
no interpretable length and are present only for graphical convenience; (c) a spot
on the dendrogram is indicated (here, by a large open circle), called the “root”. A
crucial characterization feature of an ultrametric is that the root is equidistant from
all terminal nodes. Given these interpretive changes, the ultrametric values for each
object pair can now be reconstructed by the length of the path in the tree connecting
the two relevant objects. Thus, an ultrametric is reconstructed from the lengths of
paths between objects in a tree; and the special form of a tree for an ultrametric
is one in which there exists a root that is equidistant from all terminal nodes. In
the generalization to an additive tree of Chapter 6, the condition of requiring the
existence of an equidistant root is removed, thus allowing the branches attached to
the terminal nodes to be stretched or shrunk at will.

5.3.1 \texttt{\LaTeX} Code for the Dendrogram of Figure 5.1

\begin{verbatim}
\begin{figure}
  \caption{A dendrogram (tree) representation for the
  ultrametric described in the text having VAF of .4941}
  \setlength{\unitlength}{.5pt}
  \begin{picture}(500,1000)
    \put(50,0){\makebox(0,0){0}}
    \put(100,0){\makebox(0,0){8}}
    \put(150,0){\makebox(0,0){2}}
    \put(200,0){\makebox(0,0){4}}
    \put(250,0){\makebox(0,0){9}}
    \put(300,0){\makebox(0,0){3}}
    \put(350,0){\makebox(0,0){6}}
    \put(400,0){\makebox(0,0){7}}
    \put(450,0){\makebox(0,0){5}}
    \put(500,0){\makebox(0,0){1}}
    \put(50,50){\circle{20}}
    \put(100,50){\circle{20}}
    \end{picture}
\end{figure}
\end{verbatim}
Figure 5.1. A dendrogram (tree) representation for the ultrametric described in the text having VAF of .4941.
5.3. Graphically Representing an Ultrametric

\put(150,50){\circle{20}}
\put(200,50){\circle{20}}
\put(250,50){\circle{20}}
\put(300,50){\circle{20}}
\put(350,50){\circle{20}}
\put(400,50){\circle{20}}
\put(450,50){\circle{20}}
\put(500,50){\circle{20}}
\put(550,50){\circle{20}}
\put(600,50){\circle{20}}
\put(650,50){\circle{20}}
\put(700,50){\circle{20}}
\put(750,50){\circle{20}}
\put(175,110){\circle*{20}}
\put(275,310){\circle*{20}}
\put(325,350){\circle*{20}}
\put(137.5,360){\circle*{20}}
\put(425,450){\circle*{20}}
\put(368.75,530){\circle*{20}}
\put(253.125,600){\circle*{20}}
\put(312.5,630){\circle*{20}}
\put(213.28125,780){\circle{30}}

\put(0,50){\line(0,1){800}}
\put(50,60){\line(0,1){720}}
\put(100,60){\line(0,1){300}}
\put(150,60){\line(0,1){50}}
\put(200,60){\line(0,1){50}}
\put(250,60){\line(0,1){250}}
\put(300,60){\line(0,1){250}}
\put(350,60){\line(0,1){290}}
\put(400,60){\line(0,1){390}}
\put(450,60){\line(0,1){390}}
\put(500,60){\line(0,1){570}}
\put(175,110){\line(0,1){250}}
\put(275,310){\line(0,1){40}}
\put(325,350){\line(0,1){180}}
\put(425,450){\line(0,1){80}}
\put(368.75,530){\line(0,1){70}}
\put(137.5,360){\line(0,1){240}}
\put(253.125,600){\line(0,1){30}}
\put(376.5625,630){\line(0,1){150}}
\put(150,110){\line(1,0){50}}
\put(250,310){\line(1,0){50}}
\put(275,350){\line(1,0){75}}
\put(100,360){\line(1,0){75}}
\put(400,450){\line(1,0){50}}
\put(312.5,530){\line(1,0){112.5}}
\put(137.5,600){\line(1,0){231.25}}
5.3.2 Plotting the Dendrogram with ultraplot.m

The m-file, ultraplot.m, uses several of the routines in the MATLAB Statistics Toolbox to plot the dendrogram associated with an ultrametric matrix. So, if the user has the latter Toolbox available, a graphical representation of the ultrametric can be generated directly with the syntax

```
ultraplot(ultra)
```

where ULTRA is the ultrametric matrix. A figure window opens in MATLAB displaying the appropriate tree, which can then be saved in one of the common graphics file formats and included in a printed document (we have typically used the encapsulated postscript form (*.eps)). The example below is how the tree looks obtained from number.dat and ultrafnd.m.

```
>> load number.dat
>> [find,vaf] = ultrafnd(number,randperm(10));

vaf =
```
0.4941

>> ultraplot(find)

additive_constant =

0

If there are any negative values in the input matrix ultra (as obtained, for example, when fitting multiple ultrametrics to a single proximity matrix), an additive constant equal to the negative of the minimum value in ultra is added to the off-diagonal entries in ultra before the plot is carried out.
Figure 5.2. *Dendrogram plot for the number data obtained using ultraplot.m*
Chapter 6

Additive Trees for Symmetric Proximity Data

A currently popular alternative to the use of a simple ultrametric in classification, and which might be considered a natural extension of the notion of an ultrametric, is that of an additive tree; comprehensive discussions can be found in Mirkin (1996, Chapter 7) or throughout Barthélémy and Guénoche (1991). Generalizing the earlier characterization of an ultrametric, an \( n \times n \) matrix \( A = \{a_{ij}\} \) can be called an additive tree (metric or matrix) if the three-object (or three-point) ultrametric condition is replaced by a four-object (or four-point) condition: 

\[
a_{ij} + a_{kl} \leq \max \{a_{ik} + a_{jl}, a_{il} + a_{jk}\} \text{ for } 1 \leq i, j, k, l \leq n;
\]

equivalently, for any object quadruple \( O_i, O_j, O_k, \) and \( O_l, \) the largest two values among the sums \( a_{ij} + a_{kl}, a_{ik} + a_{jl}, \) and \( a_{il} + a_{jk} \) must be equal.

Any additive tree matrix \( A \) can be represented (in many ways) as a sum of two matrices, say \( U = \{u_{ij}\} \) and \( C = \{c_{ij}\}, \) where \( U \) is an ultrametric matrix, and 

\[
c_{ij} = g_i + g_j \text{ for } 1 \leq i \neq j \leq n \text{ and } c_{ii} = 0 \text{ for } 1 \leq i \leq n,\]

based on some set of values \( g_1, \ldots, g_n \) (Carroll, Clark, & DeSarbo, 1984, pp. 71–72). The multiplicity of such possible decompositions results from the choice of where essentially to place the root in the type of graphical tree representation we will use. Generally, for us, the root will be placed half-way along the longest path in the tree, generating a decomposition of the matrix \( A \) using a procedure from Barthélémy and Guénoche (1991, Section 3.3.3):

(a) Given \( A, \) let \( O_{is}, O_{js} \in S \) denote the two objects between which the longest path is defined in the tree, i.e., the pair of objects \( O_{is} \) and \( O_{js} \) is associated with the largest entry in \( A, \) say \( a_{isjs}. \)

(b) Define \( U \) by letting 

\[
u_{ij} = a_{ij} - (g_i + g_j), \text{ where } g_i = \max\{a_{is*}, a_{j*j*}\} - M,
\]

with \( M \) chosen so that \( u_{ij} > 0 \) for \( i \neq j. \) The matrix \( C = \{c_{ij}\} \) is then constructed by letting \( c_{ii} = 0 \) for \( 1 \leq i \leq n, \) and \( c_{ij} = g_i + g_j \) for \( 1 \leq i \neq j \leq n. \) (If \( M \) is set equal to the largest entry \( a_{isjs}, \) the values in \( U \) would have to be positive, and two values among \( g_1, \ldots, g_n \) would be zero with the remainder less than or equal to zero. Thus, a value for \( M \) less than \( a_{isjs} \) is usually found by trial-and-error that will give
positive entries within \( U \) and as many positive values as possible for \( g_1, \ldots, g_n \).

To construct the type of graphical additive tree representation we will give below, the process followed is first to graph the dendrogram induced by \( U \), where (as for any ultrametric) the chosen root is equidistant from all terminal nodes. The branches connecting the terminal nodes are then lengthened or shortened depending on the signs and absolute magnitudes of \( g_1, \ldots, g_n \). If one were willing to consider the (arbitrary) inclusion of a sufficiently large additive constant to the entries of \( A \), the values of \( g_1, \ldots, g_n \) could be assumed non-negative. In this case, the matrix \( C \) would represent what is now commonly called a centroid metric (see, for example, the usage in Barthélemy and Guénoche, 1991, Chapter 3); although having some advantages (particularly for some of the graphical representations we give in avoiding the issue of presenting negative branch lengths), such a restriction is not absolutely necessary for what we do in the remaining sections and chapters of Part II. In fact, even though some of the entries among \( g_1, \ldots, g_n \) may be negative, for convenience we will still routinely refer to a centroid metric (component) even though some of the defined “distances” may actually be negative.

### 6.1 Fitting a Given Additive Tree in the \( L_2 \)-Norm

The function, \texttt{atreefit.m}, with usage

\[
\texttt{[fit,vaf] = atreefit(prox,targ)}
\]

parallels that of \texttt{ultrafit.m} of Section 5.1, and generates (again using iterative projection based on the linear (in)equality constraints obtained from a fixed additive tree — see Section 1.4) the best-fitting additive tree in the \( L_2 \)-norm (\( \text{FIT} \)) within the same equivalence class as that of a given additive tree matrix \( \text{TARG} \). Thus, both \( \text{FIT} \) and \( \text{TARG} \) satisfy the exact same set of four-point conditions. The matrix \( \text{PROX} \) contains the symmetric input proximities and \( \text{VAF} \) is the variance-accounted-for.

In the example below, the target matrix is again \texttt{numcltarg} obtained from the complete-link hierarchical clustering of \texttt{number.dat}; the \( \text{VAF} \) generated by these (now considered as additive tree) constraints is .6249 (and, as to be expected, is a value larger than for the corresponding best-fitting ultrametric value of .4781).

\[
\texttt{[fit,vaf] = atreefit(number,numcltarg)}
\]

\[
\text{fit} =
\]

\begin{table}
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\begin{tabular}{cccccc}
0 & 0.4210 & 0.7185 & 0.7371 & 0.7092 & 0.8188 \\
0.4210 & 0 & 0.5334 & 0.6520 & 0.5241 & 0.6337 \\
0.7185 & 0.5334 & 0 & 0.4882 & 0.0590 & 0.5700 \\
0.7371 & 0.5334 & 0.4882 & 0 & 0.4790 & 0.4337 \\
0.7092 & 0.5241 & 0.0590 & 0.4790 & 0 & 0.5607 \\
0.8188 & 0.6337 & 0.5700 & 0.4337 & 0.5607 & 0 \\
0.7116 & 0.5265 & 0.4652 & 0.2506 & 0.4535 & 0.4082 \\
\end{tabular}
\end{table}
6.2. Finding an Additive Tree in the $L_2$-Norm

Analogous to the m-file, `ultrafnd.m`, from Section 5.2 for identifying best-fitting ultrametrics, `atreefnd.m` implements the Hubert and Arabie (1995b) heuristic search strategy using iterative projection but now for constructing the best-fitting additive trees in the $L_2$-norm. The usage has the form

```
[find,vaf] = atreefnd(prox,inperm)
```

where `FIND` is the identified additive tree having `VAF`. Again, the matrix `PROX` contains the symmetric input proximities, and `INPERM` is a permutation that defines an order in which the constraints are considered over all object quadruples. In the example below, two such searches are shown starting with random permutations (through the use of `randperm(10)`) that give VAFs of .6359 and .6249.

```
[find,vaf] = atreefnd(number,randperm(10))
```

**find =**

Columns 1 through 6

```
0 0.4210 0.6467 0.6448 0.6374 0.8049
0.4210 0 0.4616 0.4596 0.4523 0.6198
0.6467 0.4616 0 0.3634 0.0590 0.5235
0.6448 0.4596 0.3634 0 0.3542 0.4385
0.6374 0.4523 0.0590 0.3542 0 0.5143
```
### Chapter 6. Additive Trees for Symmetric Proximity Data

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**Columns 7 through 10**

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<td>0.3941</td>
<td>0.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3400</td>
<td>0.4569</td>
<td>0.3941</td>
<td>0.0000</td>
<td>0.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \text{vaf} = 0.6359 \]

\[
\begin{align*}
\text{[find, vaf]} &= \text{atreefnd(number, randperm(10))} \\
\text{find} &= \\
\text{Columns 1 through 6} \\
0 & 0.4210 0.7185 0.7371 0.7092 0.8188 \\
0.4210 & 0 0.5334 0.5520 0.5241 0.6337 \\
0.7185 & 0.5334 0 0.4882 0.0590 0.5700 \\
0.7371 & 0.5520 0.4882 0 0.4790 0.4337 \\
0.7092 & 0.5241 0.0590 0.4790 0 0.5607 \\
0.8188 & 0.6337 0.5700 0.4337 0.5607 0 \\
0.7116 & 0.5265 0.4627 0.2560 0.4535 0.4082 \\
0.8670 & 0.6818 0.6181 0.4818 0.6089 0.4000 \\
0.7549 & 0.5698 0.3111 0.5247 0.3019 0.6064 \\
0.8318 & 0.6467 0.5830 0.2630 0.5737 0.5284 \\
\text{Columns 7 through 10} \\
0.7116 & 0.8670 0.7549 0.8318 \\
0.5265 & 0.6818 0.5698 0.6467 \\
0.4627 & 0.6181 0.3111 0.5830 \\
0.2506 & 0.4818 0.5247 0.2630 
\end{align*}
\]
6.3. Decomposing an Additive Tree

The m-file, `atreedec.m`, decomposes a given additive tree matrix into an ultrametric and a centroid metric matrix (where the root is half-way along the longest path). The form of the usage is

```
[ulmetric,ctmetric] = atreedec(prox,constant)
```

where `PROX` is the input (additive tree) proximity matrix (with a zero main diagonal and a dissimilarity interpretation); `CONSTANT` is a non-negative number (less than or equal to the maximum proximity value) that controls the positivity of the constructed ultrametric values; `ULMETRIC` is the ultrametric component of the decomposition; `CTMETRIC` is the centroid metric component (given by values $g_1, \ldots, g_n$ assigned to each of the objects, some of which may actually be negative depending on the input proximity matrix used). In the example below, the additive tree matrix identified earlier with a VAF of .6359, is decomposed using a value of .70 for the constant to control the positivity of the ultrametric values.

```
>> [find,vaf] = atreefnd(number,randperm(10));
>> [ulmetric,ctmetric] = atreedec(find,.70);
>> ulmetric

ulmetric =

Columns 1 through 6

0 0.1536 0.4737 0.4737 0.4737 0.4737
0.1536 0 0.4737 0.4737 0.4737 0.4737
0.4737 0.4737 0 0.4720 0.1749 0.4720
0.4737 0.4737 0.4720 0 0.4720 0.3888
0.4737 0.4737 0.1749 0.4720 0 0.4720
0.4737 0.4737 0.4720 0.3888 0.4720 0.2560
0.4737 0.4737 0.4720 0.3888 0.4720 0.2560
0.4737 0.4737 0.4720 0.3888 0.4720 0.2560
```
0.4737 0.4737 0.4720 0.3888 0.4720 0.2560

Columns 7 through 10

0.4737 0.4737 0.4737 0.4737
0.4737 0.4737 0.4737 0.4737
0.4720 0.4720 0.4720 0.4720
0.3888 0.3888 0.3888 0.3888
0.4720 0.4720 0.4720 0.4720
0.2560 0.2560 0.2560 0.2560
0 0.1144 0.1144 0.1144
0.1144 0 0.0103 0.0574
0.1144 0.0103 0 0.0574
0.1144 0.0574 0.0574 0

>> ctmetric'

ans =

Columns 1 through 6

0.2263 0.0412 -0.0533 -0.0552 -0.0626 0.1049

Columns 7 through 10

0.0523 0.2263 0.1634 0.1733

[orderprox,orderperm] = ultraorder(ulmetric)

orderprox =

Columns 1 through 7

0 0.1536 0.4737 0.4737 0.4737 0.4737 0.4737
0.1536 0 0.4737 0.4737 0.4737 0.4737 0.4737
0.4737 0.4737 0 0.1749 0.4720 0.4720 0.4720
0.4737 0.4737 0.1749 0 0.4720 0.4720 0.4720
0.4737 0.4737 0.4720 0.4720 0 0.2560 0.2560
0.4737 0.4737 0.4720 0.4720 0.2560 0 0.1144
0.4737 0.4737 0.4720 0.4720 0.2560 0.1144 0
0.4737 0.4737 0.4720 0.4720 0.2560 0.1144 0.0574
0.4737 0.4737 0.4720 0.4720 0.1144 0.0574 0.0574
0.4737 0.4737 0.4720 0.4720 0.3888 0.3888 0.3888

Columns 8 through 10
6.4 Graphically Representing an Additive Tree

The information present in an additive tree can be provided in several ways. First, given the decomposition into an ultrametric and a centroid metric, the partition hierarchy induced by the ultrametric could be given explicitly, along with the levels at which the various new subsets in the partitions are formed. The fitted additive tree values could then be identified as a sum of (a) the level at which an object pair, say \( O_i \) and \( O_j \), first appear together within a common subset of the hierarchy, and (b) the sum of \( g_i \) and \( g_j \) for the pair from the centroid metric component. As an illustration using the example just given in Section 6.3, the partition hierarchy has the form:

<table>
<thead>
<tr>
<th>Partition</th>
<th>Level Formed</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1,0,2,4,5,6,9,8,7,3}</td>
<td>.47</td>
</tr>
<tr>
<td>{1,0},{2,4},{5,6,9,8,7,3}</td>
<td>.39</td>
</tr>
<tr>
<td>{1,0},{2,4},{5,6,9,8,7},{3}</td>
<td>.26</td>
</tr>
<tr>
<td>{1,0},{2,4},{5},{6,9,8,7},{3}</td>
<td>.17</td>
</tr>
<tr>
<td>{1,0},{2},{4},{5},{6,9,8,7},{3}</td>
<td>.15</td>
</tr>
<tr>
<td>{1},{0},{2},{4},{5},{6,9,8,7},{3}</td>
<td>.11</td>
</tr>
<tr>
<td>{1},{0},{2},{4},{5},{6},{9,8,7},{3}</td>
<td>.06</td>
</tr>
<tr>
<td>{1},{0},{2},{4},{5},{6},{9},{8,7},{3}</td>
<td>.01</td>
</tr>
<tr>
<td>{1},{0},{2},{4},{5},{6},{9},{8},{7},{3}</td>
<td>.00</td>
</tr>
</tbody>
</table>

with centroid metric values of:
Thus, the additive tree value for the digit pair (3, 6) of .39 [.3858] is formed from the level .39 [.3888] at which 3 and 6 first appear together in the hierarchy, plus the sum of the $g_i$s for the two digits of -.06 [-.0552] and .05 [.0523]. A dendrogram representation for the partition hierarchy is given in Figure 6.1.

A graphical representation for the additive tree is given in Figure 6.2 which was obtained from the dendrogram of Figure 6.1 by stretching and shrinking the branches attached to the terminal nodes by the $g_i$ values (and cutting the vertical scale given in the dendrogram by half). Thus, the length of a path in the tree from one terminal node to another (ignoring all horizontal lines as having uninterpretable lengths), would generate the values given in the additive tree matrix.

### 6.5 An Alternative for Finding an Additive Tree in the $L_2$-Norm (Based on Combining a Centroid Metric and an Ultrametric)

If the four-point condition characterizing an additive tree is strengthened so that all the sums in the defining conditions for all object quadruples are equal (and not only for the largest two such sums), the additive tree matrix so obtained has entries representable as $g_i + g_j$, for a collection of values $g_1, \ldots, g_n$. This specially constrained additive tree is usually referred to as a centroid metric, and as noted by Carroll and Pruzansky (1980) and De Soete, DeSarbo, Furnas, and Carroll (1984), can be fitted to a proximity matrix in the $L_2$-norm through closed-form expressions. Specifically, if $P$ denotes the proximity matrix, then $g_i$ can be given as the $i^{th}$ row sum of $P$ excluding the diagonal entry, divided by $n-2$, minus the total off-diagonal sum divided by $2(n-1)(n-2)$.

The m-file, `centfit.m`, for obtaining the best-fitting centroid metric in the $L_2$-norm, has usage

```matlab
function [fit, vaf, lengths] = centfit(prox)
```

where `PROX` is the usual input proximity matrix (with a zero main diagonal and a dissimilarity interpretation); the $n$ values that define the approximating sums, $g_i + g_j$, present in the fitted matrix `FIT`, are given in the vector `LENGTHS` of size
6.5. An Alternative for Finding an Additive Tree in the $L_2$-Norm

Figure 6.1. A dendrogram (tree) representation for the ultrametric component of the additive tree described in the text having VAF of .6359
Figure 6.2. A graph-theoretic representation for the additive tree described in the text having VAF of .6359.
6.5. An Alternative for Finding an Additive Tree in the $L_2$-Norm

$n \times 1$. The example below uses centfit.m with the number.dat data set, leading to an additive tree with VAF of .3248; this tree could be represented graphically as a "star" tree with one internal node and spokes having the lengths given in the output vector LENGTHS.

load number.dat
[fit,vaf,lengths] = centfit(number)

fit =

Columns 1 through 7

0 0.7808 0.6877 0.6709 0.6784 0.7647 0.6589
0.7808 0 0.5026 0.4858 0.4933 0.5796 0.4738
0.6877 0.5026 0 0.3927 0.4002 0.4864 0.3807
0.6709 0.4858 0.3927 0 0.3834 0.4697 0.3639
0.6784 0.4933 0.4002 0.3834 0 0.4772 0.3714
0.7647 0.5796 0.4864 0.4697 0.4772 0 0.4577
0.6589 0.4738 0.3807 0.3639 0.3714 0.4577 0
0.8128 0.6277 0.5346 0.5178 0.5253 0.6116 0.5058
0.7499 0.5648 0.4717 0.4549 0.4624 0.5487 0.4429
0.7657 0.5806 0.4874 0.4707 0.4782 0.5644 0.4587

Columns 8 through 10

0.8128 0.7499 0.7657
0.6277 0.5648 0.5806
0.5346 0.4717 0.4874
0.5178 0.4549 0.4707
0.5253 0.4624 0.4782
0.6116 0.5487 0.5644
0.5058 0.4429 0.4587
0 0.5968 0.6126
0.5968 0 0.5497
0.6126 0.5497 0

vaf =

0.3248

lengths =

Columns 1 through 7

0.4830 0.2978 0.2047 0.1880 0.1955 0.2817 0.1760
Columns 8 through 10

0.3298  0.2670  0.2827

An alternative strategy for identifying good-fitting additive trees (and one that will be used in a slightly different form on two-mode proximity data in Section 8.2) relies on the possible decomposition of an additive tree into an ultrametric and centroid metric. The m-file, atreectul.m, first fits a centroid metric in closed form; an ultrametric is then identified on the residual matrix. The sum of these two matrices is an additive tree. The usage would follow that of atreefnd.m:

\[
[f\text{ind},v\text{af}] = \text{atreectul}(\text{prox},\text{inperm})
\]

where FIND is the identified additive tree with VAF. Again, the matrix PROX contains the symmetric input proximities, and INPERM is a permutation that defines an order in which the constraints are considered over all object triples in the identification of the ultrametric component. In the example below, one search is shown starting with a random permutation (through the use of \text{randperm(10)}) that gives the same additive tree identified earlier with a VAF of .6249.

\[
>> [f\text{ind},v\text{af}] = \text{atreectul}(\text{number},\text{randperm(10)});
\]

\[
>> v\text{af}
\]

vaf =

0.6249
Chapter 7

Fitting Multiple Tree Structures to a Symmetric Proximity Matrix

The use of multiple structures, whether they be ultrametrics or additive trees, to represent additively a given proximity matrix, proceeds directly through successive residualization and iteration. We restrict ourselves to the fitting of two such structures but the same process would apply for any such number. Initially, a first matrix is fitted to a given proximity matrix and a first residual matrix obtained; a second structure is then fitted to these first residuals, producing a second residual matrix. Iterating, the second fitted matrix is now subtracted from the original proximity matrix and a first (re)fitted matrix obtained; this first (re)fitted matrix in turn is subtracted from the original proximity matrix and a new second matrix (re)fitted. This process continues until the \( VAF \) by the sum of both fitted matrices no longer changes by a set amount (the value of \( 1.0e-006 \) is used in the m-files of the next two sections).

7.1 Multiple Ultrametrics

The m-file, biultrafnd.m, fits (additively) two ultrametric matrices in the \( L_2 \)-norm. The explicit usage is

\[
\text{[find,vaf,targone,targtwo] = biultrafnd(prox,inperm)}
\]

where PROX is the given input proximity matrix (with a zero main diagonal and a dissimilarity interpretation); INPERM is a permutation that determines the order in which the inequality constraints are considered (and thus can be made random to search for different locally optimal representations); FIND is the obtained least-squares matrix (with variance-accounted-for of VAF) to PROX, and is the sum of the two ultrametric matrices TARGONE and TARGTWO.

In the example to follow, a VAF of .8001 was achieved for the two identified ultrametrics (and where one needs to add an (arbitrary) constant [e.g., .40] to the entries in TARGTWO to satisfy the technical requirement here that ultrametric values should be non-negative). It might be noted substantively that the first ultrametric matrix (in TARGONE) reflects the structural properties of the digits; the second ul-
Chapter 7. Fitting Multiple Tree Structures for a Symmetric Matrix

Ultrametric matrix (in TARGTWO) is completely consistent with digit magnitude. This result is a very nice mixture of ultrametric structures with a convenient substantive interpretation for both components.

```matlab
>> [find, vaf, targone, targtwo] = biultrafnd(number, randperm(10));
```

```matlab
>> vaf

vaf =

0.8001
```

```matlab
>> [orderproxone, orderpermone] = ultraorder(targone)
```

```matlab
orderproxone =

Columns 1 through 6

0    0.7796  0.7796  0.7796  0.7796  0.7796
0.7796    0  0.2168  0.2168  0.5512  0.5512
0.7796  0.2168    0  0.0701  0.5512  0.5512
0.7796  0.2168  0.0701    0  0.5512  0.5512
0.7796  0.5512  0.5512  0.5512  0    0.1733
0.7796  0.5512  0.5512  0.5512  0.1733  0

Columns 7 through 10

0.7796  0.7796  0.7796  0.7796
0.5512  0.5512  0.5512  0.5945
0.5512  0.5512  0.5512  0.5945
0.5512  0.5512  0.5512  0.5945
0.2772  0.4622  0.4622  0.5945
0.2772  0.4622  0.4622  0.5945
0    0.4622  0.4622  0.5945
0.4622    0  0.3103  0.5945
0.4622  0.3103    0  0.5945
0.5945  0.5945  0.5945  0
```

```matlab
orderpermone =

1    9    3    5   10    4    7    8    6    2
```
>> [orderproxtwo,orderpermtwo] = ultraorder(targtwo)

orderproxtwo =

Columns 1 through 6

0 -0.3586 -0.2531 -0.1721 -0.0111 -0.0111
-0.3586 0 -0.2531 -0.1721 -0.0111 -0.0111
-0.2531 -0.2531 0 -0.1721 -0.0111 -0.0111
-0.1721 -0.1721 -0.1721 0 -0.0111 -0.0111
-0.0111 -0.0111 -0.0111 -0.0111 0 -0.1422
-0.0111 -0.0111 -0.0111 -0.0111 -0.1422 0
0.0897 0.0897 0.0897 0.0897 0.0897 0.0897
0.0897 0.0897 0.0897 0.0897 0.0897 0.0897
0.0897 0.0897 0.0897 0.0897 0.0897 0.0897
0.0897 0.0897 0.0897 0.0897 0.0897 0.0897

Columns 7 through 10

0.0897 0.0897 0.0897 0.0897
0.0897 0.0897 0.0897 0.0897
0.0897 0.0897 0.0897 0.0897
0.0897 0.0897 0.0897 0.0897
0.0897 0.0897 0.0897 0.0897
0.0897 0.0897 0.0897 0.0897
0 0.0982 0.0982 -0.0479
-0.0982 0 0.2012 -0.0479
-0.0982 -0.2012 0 -0.0479
-0.0479 -0.0479 -0.0479 0

orderpermtwo =

1 2 3 4 5 6 8 7 9 10

7.2 Multiple Additive Trees

The m-file, biatreefnd.m, fits (additively) two additive tree matrices in the $L_2$-norm. The explicit usage is

[find,vaf,targone,targtwo] = biatreefnd(prox,inperm)

where PROX is the given input proximity matrix (with a zero main diagonal and a dissimilarity interpretation); INPERM is a permutation that determines the order in which the inequality constraints are considered (and thus can be made random to search for different locally optimal representations); FIND is the found least-squares...
matrix (with variance-accounted-for of VAF) to PROX, and is the sum of the two additive tree matrices TARGONE and TARGTWO.

In the example to follow, a VAF of .9003 was achieved for the two identified additive trees (and where one needs as in the multiple ultrametric case, to add an [arbitrary] constant to the entries in TARGTWO to satisfy the technical requirement here that additive tree values should be non-negative; also, sufficiently large additive constants would need to be imposed on the two ultrametric components to ensure non-negativity of the resulting values. These additive constants do not affect the achieved VAF or the resulting representations in any material way.). Similarly, as in the interpretation for the example of the last section, it might be noted substantively that the second additive tree matrix (in TARGTWO) reflects the structural properties of the digits; the first matrix (in TARGONE) is completely consistent with digit magnitude. So, again we have a very nice mixture of structures with convenient substantive interpretations for both components.

```matlab
>> [find,vaf,targone,targtwo] = biatreefnd(number,randperm(10));

>> vaf
vaf =
    0.9003

>> [ulmetricone,ctmetricone] = atreedec(targone,0.0);
>> [ulmetrictwo,ctmetrictwo] = atreedec(targtwo,0.0);
>> ctmetricone'
ans =
     Columns 1 through 6
          0.9652     0.7801     0.6716     0.6164     0.7114     0.7976
     Columns 7 through 10
          0.7699     0.9652     0.9023     0.9051

>> ctmetrictwo'
ans =
     Columns 1 through 6
          0.0373     0.0994     0.1256     0.1256     0.1256     0.1129
     Columns 7 through 10
```
7.2. Multiple Additive Trees

0.0267 0.1129 0.1105 0.1256

>> [orderproxone,orderpermone] = ultraorder(ulmetricone)

orderproxone =

Columns 1 through 6

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<td>-0.9652</td>
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<td>0</td>
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<td>-0.9652</td>
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</tr>
<tr>
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<td>-0.9652</td>
<td>-1.1030</td>
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</tr>
</tbody>
</table>

Columns 7 through 10

<table>
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<th>-0.9652</th>
<th>-0.9652</th>
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</tr>
</thead>
<tbody>
<tr>
<td>-0.9652</td>
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<td>-0.9652</td>
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<tr>
<td>-1.1030</td>
<td>-1.1030</td>
<td>-1.1030</td>
<td>-0.9850</td>
</tr>
<tr>
<td>-1.1030</td>
<td>-1.1030</td>
<td>-1.1030</td>
<td>-0.9850</td>
</tr>
<tr>
<td>-1.4617</td>
<td>-1.4617</td>
<td>-1.3477</td>
<td>-0.9850</td>
</tr>
<tr>
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<td>-1.3477</td>
<td>0</td>
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<tr>
<td>-0.9850</td>
<td>-0.9850</td>
<td>-0.9850</td>
<td>0</td>
</tr>
</tbody>
</table>

orderpermone =

3 2 1 5 6 10 9 8 7 4

>> [orderproxtwo,orderpermtwo] = ultraorder(ulmetrictwo)

orderproxtwo =

Columns 1 through 6

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<td>-0.4893</td>
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</table>
### Chapter 7. Fitting Multiple Tree Structures for a Symmetric Matrix

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<td>-0.1256</td>
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Columns 7 through 10

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<td>-0.3151</td>
<td>-0.1596</td>
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<td>-0.3151</td>
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<td>-0.1596</td>
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<tr>
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<td>-0.1596</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

orderperm\(two\) =

```
7  9  5  3  8  6  10  4  2  1
```
Chapter 8

Ultrametrics and Additive Trees for Two-Mode (Rectangular) Proximity Data

Thus far in this Part II, the proximity data considered for obtaining some type of structure, such as an ultrametric or an additive tree, have been assumed to be on one intact set of objects, \( S = \{ O_1, \ldots, O_n \} \), and complete in the sense that proximity values are present between all object pairs. Just as LUS (Linear Unidimensional Scaling) was generalized for two-mode proximity data in Chapter 4, suppose now that the available proximity data are two-mode, and between two distinct object sets, \( S_A = \{ O_{1A}, \ldots, O_{n_A} \} \) and \( S_B = \{ O_{1B}, \ldots, O_{n_B} \} \), containing \( n_A \) and \( n_B \) objects, respectively, given by an \( n_A \times n_B \) proximity matrix \( Q = \{ q_{rs} \} \). Again, we assume that the entries in \( Q \) are keyed as dissimilarities, and a joint structural representation is desired for the set \( S_A \cup S_B \).

Conditions have been proposed in the literature for when the entries in a matrix fitted to \( Q \) characterize an ultrametric or an additive tree representation. In particular, suppose a \( n_A \times n_B \) matrix \( F = \{ f_{rs} \} \) is fitted to \( Q \) through least squares subject to the constraints that follow:

**Ultrametric** (Furnas, 1980):
for all distinct object quadruples, \( O_{rA}, O_{sA}, O_{rB}, O_{sB} \), where \( O_{rA}, O_{sA} \in S_A \) and \( O_{rB}, O_{sB} \in S_B \), and considering the entries in \( F \) corresponding to the pairs, \( (O_{rA}, O_{rB}), (O_{sA}, O_{sB}) \), and \( (O_{sA}, O_{rB}) \), say \( f_{r_A r_B}, f_{r_A s_B}, f_{s_A r_B}, f_{s_A s_B} \), respectively, the largest two must be equal.

**Additive trees** (Brossier, 1987):
for all distinct object sextuples, \( O_{rA}, O_{sA}, O_{tA}, O_{rB}, O_{sB}, O_{tB} \), where \( O_{rA}, O_{sA}, O_{tA} \in S_A \) and \( O_{rB}, O_{sB}, O_{tB} \in S_B \), and considering the entries in \( F \) corresponding to the pairs \( (O_{rA}, O_{rB}), (O_{sA}, O_{sB}), (O_{sA}, O_{tB}), (O_{sA}, O_{rB}), (O_{sA}, O_{tA}), (O_{sA}, O_{tB}), (O_{sA}, O_{sB}), (O_{tA}, O_{rB}), (O_{tA}, O_{sB}), \) and \( (O_{tA}, O_{tB}) \), say \( f_{r_A r_B}, f_{r_A s_B}, f_{r_A t_B}, f_{s_A s_B}, f_{s_A t_B}, f_{s_A r_B}, f_{s_A t_B}, f_{s_A s_B}, f_{t_A s_B}, f_{t_A t_B}, f_{t_A r_B}, f_{t_A t_B} \), respectively, the largest two of the following sums must be equal:

\[
\begin{align*}
&f_{r_A r_B} + f_{s_A s_B} + f_{t_A t_B}; \\
&f_{r_A r_B} + f_{s_A t_B} + f_{t_A s_B};
\end{align*}
\]
In both cases of ultrametric and additive trees for two-mode proximity data, the necessary constraints characterizing a solution are linear and define closed convex sets in which a solution must lie. Thus, the application of iterative projection as a heuristic search strategy for the best-fitting solutions is fairly direct, and an example of an ultrametric found and fitted to a two-mode matrix will be given in Section 8.1. We will not, however, give a comparable example of fitting the additive tree constraints to such a proximity matrix; the (scratch) storage requirements necessitated by iterative projection in directly using the additive tree constraints given above and keeping track of the various augmentations made can become rather onerous for moderate-sized data matrices in the course of the heuristic search. For general use, an alternative approach to the fitting of additive trees is preferable that again uses iterative projection but with the ultrametric conditions in conjunction with a secondary centroid metric; this strategy avoids any major (scratch) storage difficulties and will be reviewed and illustrated in Section 8.2. We might note that the process of fitting two-mode proximity data by additive trees or ultrametrics using iterative projection heuristics may generate a rather large number of distinct locally optimal solutions with differing VAF values, particularly in contrast to the situation usually observed for symmetric proximity data. Although this abundance is not inevitably the case and obviously depends on the particular data set being considered, it is not unusual and should be expected by a user.

8.1 Fitting and Finding Two-Mode Ultrametrics

To illustrate the fitting of a given two-mode ultrametric, a two-mode target is generated by the upper-right $6 \times 4$ portion of the $10 \times 10$ ultrametric target matrix, numcltarg, used in Section 5.1. This file will be called numcltarg6x4.dat, and has contents as follows:

```
9 9 9 9
9 9 9 9
8 8 4 8
3 7 8 2
8 8 4 8
7 5 8 7
```

The six rows correspond to the digits 0, 1, 2, 3, 4, and 5; the four columns to 6, 7, 8, and 9. As the two-mode $6 \times 4$ proximity matrix, the appropriate upper-right portion of the number proximity matrix will be used in the fitting process; the corresponding file is called number6x4.dat, with contents:

```
.788 .909 .821 .850
.758 .630 .791 .625
```
8.1. Fitting and Finding Two-Mode Ultrametrics

The m-file, `ultrafittm.m`, fits a given ultrametric to a two-mode proximity matrix (using iterative projection in the $L_2$-norm), and has usage:

```
[fit, vaf] = ultrafittm(proxtm, targ)
```

where `PROXTM` is the two-mode (rectangular) input proximity matrix (with a dissimilarity interpretation); `TARG` is an ultrametric matrix of the same size as `PROXTM`; `FIT` is the least-squares optimal matrix (with variance-accounted-for of `VAF`) to `PROXTM` satisfying the two-mode ultrametric constraints implicit in `TARG`. An example follows using `numcltarg6x4` for `TARG` and `number6x4` as `PROXTM`:

```matlab
>> load number6x4.dat
>> load numcltarg6x4.dat
>> [fit, vaf] = ultrafittm(number6x4, numcltarg6x4)
```

```
fit =

0.7715 0.7715 0.7715 0.7715
0.7715 0.7715 0.7715 0.7715
0.6641 0.6641 0.3065 0.6641
0.3000 0.5267 0.6641 0.2630
0.6641 0.6641 0.3065 0.6641
0.5267 0.4000 0.6641 0.5267

vaf =

0.6978
```

A VAF of .6978 was obtained for the fitted ultrametric; we give the hierarchy below with indications of when the partitions were formed in the $L_2$-norm fitted ultrametric (in `FIT`) and in the original target (in `cltarg6X4`):

<table>
<thead>
<tr>
<th>Partition</th>
<th>Level Formed $(L_2)$</th>
<th>Level Formed (Target)</th>
</tr>
</thead>
<tbody>
<tr>
<td>{0,1,2,4,8,3,9,6,5,7}</td>
<td>.7715</td>
<td>9</td>
</tr>
<tr>
<td>{0}, {1}, {2,4,8,3,9,6,5,7}</td>
<td>.6641</td>
<td>8</td>
</tr>
<tr>
<td>{0}, {1}, {2,4,8,3,9,6,5,7}</td>
<td>.5267</td>
<td>7</td>
</tr>
<tr>
<td>{0}, {1}, {2,4,8,3,9,6}, {5}, {7}</td>
<td>.4000</td>
<td>5</td>
</tr>
<tr>
<td>{0}, {1}, {2,4,8,3,9,6}, {5}, {7}</td>
<td>.3065</td>
<td>4</td>
</tr>
<tr>
<td>{0}, {1}, {2}, {4}, {8}, {3,9,6}, {5}, {7}</td>
<td>.3000</td>
<td>3</td>
</tr>
<tr>
<td>{0}, {1}, {2}, {4}, {8}, {3,9}, {6}, {5}, {7}</td>
<td>.2630</td>
<td>2</td>
</tr>
<tr>
<td>{0}, {1}, {2}, {4}, {8}, {3}, {9}, {6}, {5}, {7}</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>
The m-file, `ultrafndtm.m`, relies on iterative projection heuristically to locate a best-fitting two-mode ultrametric. The usage is

```
[find,vaf] = ultrafndtm(proxtm,inpermrow,inpermcol)
```

where `PROXTM` is the two-mode input proximity matrix (with a dissimilarity interpretation); `INPERMROW` and `INPERMCOL` are permutations for the row and column objects that determine the order in which the inequality constraints are considered; `FIND` is the found least-squares matrix (with variance-accounted-for of `VAF`) to `PROXTM` satisfying the ultrametric constraints. The example below for the `number6x4` two-mode data (using random permutations for `INPERMROW` and `INPERMCOL`), finds an ultrametric with `VAF` of .7448.

```
[find,vaf] = ultrafndtm(number6x4,randperm(6),randperm(4))
```

```
find =
0.8420 0.8420 0.8420 0.8420
0.7010 0.7010 0.7010 0.7010
0.6641 0.6641 0.3670 0.6641
0.3000 0.5267 0.6641 0.2630
0.6641 0.6641 0.2460 0.6641
0.5267 0.4000 0.6641 0.5267
```

```
vaf =
0.7448
```

The partition hierarchy identified is similar to that found for the fixed target `numcltarg6x4`, although there is some minor variation in how the digits 0 and 1 are treated:

<table>
<thead>
<tr>
<th>Partition</th>
<th>Level Formed ((L_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>{0,1,2,4,8,3,9,6,5,7}</td>
<td>.8420</td>
</tr>
<tr>
<td>{0},{1,2,4,8,3,9,6,5,7}</td>
<td>.7010</td>
</tr>
<tr>
<td>{0},{1},{2,4,8,3,9,6,5,7}</td>
<td>.6641</td>
</tr>
<tr>
<td>{0},{1},{2,4,8},{3,9,6,5,7}</td>
<td>.5267</td>
</tr>
<tr>
<td>{0},{1},{2,4,8},{3,9},{6,5,7}</td>
<td>.4000</td>
</tr>
<tr>
<td>{0},{1},{2,4,8},{3,9},{5},{7}</td>
<td>.3670</td>
</tr>
<tr>
<td>{0},{1},{2},{4,8},{3,9},{5},{7}</td>
<td>.3000</td>
</tr>
<tr>
<td>{0},{1},{2},{4,8},{3,9},{6},{5,7}</td>
<td>.2630</td>
</tr>
<tr>
<td>{0},{1},{2},{4,8},{3},{9},{6},{5},{7}</td>
<td>.2460</td>
</tr>
<tr>
<td>{0},{1},{2},{4},{8},{3},{9},{6},{5},{7}</td>
<td>—</td>
</tr>
</tbody>
</table>

### 8.2 Finding Two-Mode Additive Trees

As noted in the introductory material to the current Chapter 8, the identification of a best-fitting two-mode additive tree will be done somewhat differently (because of
8.2. Finding Two-Mode Additive Trees

storage considerations) than for a two-mode ultrametric representation. Specifically, a (two-mode) centroid metric and a (two-mode) ultrametric matrix will be identified so that their sum is a good-fitting two-mode additive tree. Because a centroid metric can be obtained in closed-form, we first illustrate the fitting of just a centroid metric to a two-mode proximity matrix with the m-file, centfittm.m. Its usage is of the form

\[ \text{[fit, vaf, lengths]} = \text{centfittm(proxtm)} \]

which gives the least-squares fitted two-mode centroid metric (FIT) to PROXTM, the two-mode rectangular input proximity matrix (with a dissimilarity interpretation). The \( n \) values (where \( n = \text{number of rows}(n_a) + \text{number of columns}(n_b) \)) serve to define the approximating sums, \( u_r + v_s \), where the \( u_r \) are for the \( n_a \) rows and the \( v_s \) for the \( n_b \) columns; these \( u_r \) and \( v_s \) values are given in the vector LENGTHS of size \( n \times 1 \), with row values first followed by the column values. The closed-form formula used for \( u_r \) (or \( v_s \)) can be given simply as the \( r^{th} \) row (or \( s^{th} \) column) mean of PROXTM minus one-half the grand mean (see Carroll & Pruzansky, 1980, and De Soete et al., 1984, for a further discussion). In the example given below using the two-mode matrix, number6x4, a two-mode centroid metric by itself has a VAF of .4737.

\[
\text{>> [fit, vaf, lengths]} = \text{centfittm(number6x4)}; \\
\text{>> fit}
\]

\[
\begin{align*}
0.7405 & \quad 0.9101 & \quad 0.8486 & \quad 0.8688 \\
0.5995 & \quad 0.7691 & \quad 0.7076 & \quad 0.7278 \\
0.4965 & \quad 0.6661 & \quad 0.6046 & \quad 0.6248 \\
0.3882 & \quad 0.5579 & \quad 0.4964 & \quad 0.5165 \\
0.4132 & \quad 0.5829 & \quad 0.5214 & \quad 0.5415 \\
0.4132 & \quad 0.5829 & \quad 0.5214 & \quad 0.5415
\end{align*}
\]

\[
\text{>> vaf}
\]

\[
\text{vaf =}
\]

\[
0.4737
\]

\[
\text{>> lengths'}
\]

\[
\text{ans =}
\]

Columns 1 through 6

\[
\begin{align*}
0.5370 & \quad 0.3960 & \quad 0.2930 & \quad 0.1847 & \quad 0.2097 & \quad 0.2097
\end{align*}
\]
Columns 7 through 10

0.2035  0.3731  0.3116  0.3318

The finding of a two-mode additive tree with the m-file, \texttt{atreefndtm.m}, proceeds iteratively. A two-mode centroid metric is first found and the original two-mode proximity matrix residualized; a two-mode ultrametric is then identified for the residual matrix. The process repeats with the centroid and ultrametric components alternatingly being refit until a small change in the overall VAF occurs (a value less than 1.0e-006 is used). The m-file has the explicit usage

\[
\text{[find,vaf,ultra,lengths] = atreefndtm(proxtm,inpermrow,inpermcol)}
\]

and as noted above, relies on iterative projection heuristically to find a two-mode ultrametric component that is added to a two-mode centroid metric to produce a two-mode additive tree. Here, \texttt{PROXTM} is the rectangular input proximity matrix (with a dissimilarity interpretation); \texttt{INPERMROW} and \texttt{INPERMCOL} are permutations for the row and column objects that determine the order in which the inequality constraints are considered; \texttt{FIND} is the found least-squares matrix (with variance-accounted-for of \texttt{VAF}) to \texttt{PROXTM} satisfying the two-mode additive tree constraints. The vector \texttt{LENGTHS} contains the row followed by column values for the two-mode centroid metric component; \texttt{ULTRA} is the ultrametric component. In the example given below, the identified two-mode additive-tree for \texttt{number6x4} has a VAF of .9053, with a nice structural interpretation of the digits along with some indication now of odd and even digit groupings. The partition hierarchy is reported below the MATLAB output along with an indication of when the various partitions are formed.

\[
\text{>> [find,vaf,ultra,lengths] = ... atreefndtm(number6x4,randperm(6),randperm(4))}
\]

\[
\text{>> find}
\]

\[
\text{find =}
\]

\[
\begin{align*}
0.6992 & \quad 0.9029 & \quad 0.9104 & \quad 0.8561 \\
0.6298 & \quad 0.6300 & \quad 0.8411 & \quad 0.7029 \\
0.4398 & \quad 0.8160 & \quad 0.3670 & \quad 0.7692 \\
0.4549 & \quad 0.5748 & \quad 0.6661 & \quad 0.2630 \\
0.3692 & \quad 0.7453 & \quad 0.2460 & \quad 0.6985 \\
0.4582 & \quad 0.4000 & \quad 0.6694 & \quad 0.5313
\end{align*}
\]

\[
\text{>> vaf}
\]

\[
\text{vaf =}
\]

\[
0.9053
\]
8.3 Completing a Two-Mode Ultrametric to one Defined on $S_A \cup S_B$

\[\begin{align*}
\text{>> ultra} \\
\text{ultra} &= \begin{bmatrix}
0.1083 & 0.0520 & 0.1083 & 0.0520 \\
0.1083 & -0.1516 & 0.1083 & -0.0318 \\
-0.0078 & 0.1083 & -0.2919 & 0.1083 \\
0.1083 & -0.0318 & 0.1083 & -0.2968 \\
-0.0078 & 0.1083 & -0.3422 & 0.1083 \\
0.1083 & -0.2099 & 0.1083 & -0.0318
\end{bmatrix} \\
\text{>> lengths'} \\
\text{ans} &= \begin{bmatrix}
0.4570 & 0.3876 & 0.3138 & 0.2127 & 0.2431 & 0.2160 \\
0.1339 & 0.3939 & 0.3451 & 0.3471
\end{bmatrix}
\end{align*}\]

<table>
<thead>
<tr>
<th>Partition</th>
<th>Level Formed</th>
</tr>
</thead>
<tbody>
<tr>
<td>${{6,4,8,2,9,3,5,7,1,0}}$</td>
<td>.1083</td>
</tr>
<tr>
<td>${{6,4,8,2},{9,3,5,7,1,0}}$</td>
<td>.0520</td>
</tr>
<tr>
<td>${{6,4,8,2},{9,3,5,7,1},{0}}$</td>
<td>-.0078</td>
</tr>
<tr>
<td>${{6},{4,8,2},{9,3,5,7,1},{0}}$</td>
<td>-.0318</td>
</tr>
<tr>
<td>${{6},{4,8,2},{9,3},{5,7,1},{0}}$</td>
<td>-.1516</td>
</tr>
<tr>
<td>${{6},{4,8,2},{9,3},{5,7},{1},{0}}$</td>
<td>-.2099</td>
</tr>
<tr>
<td>${{6},{4,8,2},{9,3},{5},{7},{1},{0}}$</td>
<td>-.2919</td>
</tr>
<tr>
<td>${{6},{4,8},{2},{9,3},{5},{7},{1},{0}}$</td>
<td>-.2968</td>
</tr>
<tr>
<td>${{6},{4,8},{2},{9},{3},{5},{7},{1},{0}}$</td>
<td>-.3422</td>
</tr>
<tr>
<td>${{6},{4},{8},{2},{9},{3},{5},{7},{1},{0}}$</td>
<td>—</td>
</tr>
</tbody>
</table>

8.3 Completing a Two-Mode Ultrametric to one Defined on $S_A \cup S_B$

Instead of relying only on our general intuition (and problem-solving skills) to transform a fitted two-mode ultrametric to one we could interpret directly as a sequence of partitions for the joint set $S_A \cup S_B$, the m-file, `ultracomptm.m`, provides the explicit completion of a given two-mode ultrametric matrix to a symmetric proximity matrix (defined on $S_A \cup S_B$ and satisfying the usual ultrametric constraints). Thus,
this completion, in effect, estimates the (missing) ultrametric values that must be present between objects from the same cluster but from the different modes. The general syntax has the form

\[ \text{[ultracomp]} = \text{ultracomptm(ultraproxtm)} \]

where \text{ULTRAPROXTM} is the \( n_a \times n_b \) fitted two-mode ultrametric matrix; \text{ULTRACOMPTM} is the completed \( n \times n \) proximity matrix having the usual ultrametric pattern for the complete object set of size \( n = n_a + n_b \). As seen in the examples below, the use of \text{ultrafndtm.m} plus \text{ultracomptm.m} on the \text{number6x4} data, and the subsequent application of the \text{ultraorder.m} routine leads directly to the partition hierarchy we identified earlier:

\[
\begin{align*}
&\text{load number6x4.dat} \\
&\text{[find,vaf] = ultrafndtm(number6x4,randperm(6),randperm(4));} \\
&\text{vaf} \\
&\text{vaf =} \\
&\quad 0.7448 \\
&\text{[ultracomp] = ultracomptm(find)}
\end{align*}
\]

\text{ultracomp =}

Columns 1 through 7

\[
\begin{array}{cccccccc}
0 & 0.8420 & 0.8420 & 0.8420 & 0.8420 & 0.8420 & 0.8420 & 0.8420 \\
0.8420 & 0 & 0.7010 & 0.7010 & 0.7010 & 0.7010 & 0.7010 & 0.7010 \\
0.8420 & 0.7010 & 0 & 0.6641 & 0.3670 & 0.6641 & 0.6641 & 0.6641 \\
0.8420 & 0.7010 & 0.6641 & 0 & 0.6641 & 0.5267 & 0.3000 & 0.6641 \\
0.8420 & 0.7010 & 0.6641 & 0.5267 & 0.6641 & 0 & 0.6641 & 0.6641 \\
0.8420 & 0.7010 & 0.6641 & 0.3670 & 0.6641 & 0 & 0.6641 & 0.5267 \\
0.8420 & 0.7010 & 0.5267 & 0.6641 & 0 & 0.3000 & 0.6641 & 0.5267 \\
0.8420 & 0.7010 & 0.6641 & 0.6641 & 0.3670 & 0.6641 & 0.2630 & 0.6641 \\
0.8420 & 0.7010 & 0.6641 & 0.6641 & 0.5267 & 0.2630 & 0.6641 & 0.5267 & 0.3000
\end{array}
\]

Columns 8 through 10

\[
\begin{array}{cccc}
0.8420 & 0.8420 & 0.8420 \\
0.7010 & 0.7010 & 0.7010 \\
0.6641 & 0.3670 & 0.6641 \\
0.5267 & 0.6641 & 0.2630 \\
0.6641 & 0.2460 & 0.6641 \\
0.4000 & 0.6641 & 0.5267 \\
0.5267 & 0.6641 & 0.3000
\end{array}
\]
8.3. Completing a Two-Mode Ultrametric to one Defined on $S_A \cup S_B$

\[
\begin{array}{ccc}
0 & 0.6641 & 0.5267 \\
0.6641 & 0 & 0.6641 \\
0.5267 & 0.6641 & 0 \\
\end{array}
\]

>> [orderprox,orderperm] = ultraorder(ultracomplement)

orderprox =

Columns 1 through 7

\[
\begin{array}{cccccccc}
0 & 0.8420 & 0.8420 & 0.8420 & 0.8420 & 0.8420 & 0.8420 \\
0.8420 & 0 & 0.2460 & 0.3670 & 0.6641 & 0.6641 & 0.6641 \\
0.8420 & 0.2460 & 0 & 0.3670 & 0.6641 & 0.6641 & 0.6641 \\
0.8420 & 0.3670 & 0.3670 & 0 & 0.6641 & 0.6641 & 0.6641 \\
0.8420 & 0.6641 & 0.6641 & 0.6641 & 0 & 0.3000 & 0.3000 \\
0.8420 & 0.6641 & 0.6641 & 0.6641 & 0.3000 & 0 & 0.2630 \\
0.8420 & 0.6641 & 0.6641 & 0.6641 & 0.3000 & 0.2630 & 0 \\
0.8420 & 0.6641 & 0.6641 & 0.6641 & 0.5267 & 0.5267 & 0.5267 \\
0.8420 & 0.6641 & 0.6641 & 0.6641 & 0.5267 & 0.5267 & 0.5267 \\
0.8420 & 0.7010 & 0.7010 & 0.7010 & 0.7010 & 0.7010 & 0.7010 \\
\end{array}
\]

Columns 8 through 10

\[
\begin{array}{cccc}
0.8420 & 0.8420 & 0.8420 \\
0.6641 & 0.6641 & 0.7010 \\
0.6641 & 0.6641 & 0.7010 \\
0.6641 & 0.6641 & 0.7010 \\
0.5267 & 0.5267 & 0.7010 \\
0.5267 & 0.5267 & 0.7010 \\
0.5267 & 0.5267 & 0.7010 \\
0 & 0.4000 & 0.7010 \\
0.4000 & 0 & 0.7010 \\
0.7010 & 0.7010 & 0 \\
\end{array}
\]

orderperm =

\[
\begin{array}{cccccccc}
1 & 9 & 5 & 3 & 7 & 10 & 4 & 8 & 6 & 2 \\
\end{array}
\]

Similarly, for the two-mode additive tree example, we have the partition hierarchy we gave initially, and what was retrieved immediately from the use of ultracomplement.m and ultraorder.m on the output ultrametric matrix, ultra:

>> [find,vaf,ultra,lengths] = ...
atreefndtm(number6x4,randperm(6),randperm(4));

>> vaf
vaf =

0.9053

>> [ultracomp] = ultracomptm(ultra)

ultracomp =

Columns 1 through 7

0 0.0520 0.1083 0.0520 0.1083 0.0520 0.1083
0.0520 0 0.1083 -0.0318 0.1083 -0.1516 0.1083
0.1083 0.1083 0 0.1083 -0.2919 0.1083 -0.0078
0.0520 -0.0318 0.1083 0 0.1083 -0.0318 0.1083
0.1083 0.1083 -0.2919 0.1083 0 0.1083 -0.0078
0.0520 -0.1516 0.1083 -0.0318 0.1083 0 0.1083
0.1083 0.1083 -0.2919 0.1083 -0.0078 0.1083 0
0.0520 -0.1516 0.1083 -0.0318 0.1083 -0.2099 0.1083
0.1083 0.1083 -0.2919 0.1083 -0.3422 0.1083 -0.0078
0.0520 -0.0318 0.1083 -0.2968 0.1083 -0.0318 0.1083

Columns 8 through 10

0.0520 0.1083 0.0520
-0.1516 0.1083 -0.0318
0.1083 -0.2919 0.1083
-0.0318 0.1083 -0.2968
0.1083 -0.3422 0.1083
-0.2099 0.1083 -0.0318
0.1083 -0.0078 0.1083
0 0.1083 -0.0318
0.1083 0 0.1083
-0.0318 0.1083 0

>> [orderprox,orderperm] = ultraorder(ultracomp)

orderprox =

Columns 1 through 7

0 -0.0078 -0.0078 -0.0078 0.1083 0.1083 0.1083
-0.0078 0 -0.3422 -0.2919 0.1083 0.1083 0.1083
-0.0078 -0.3422 0 -0.2919 0.1083 0.1083 0.1083
-0.0078 -0.2919 -0.2919 0 0.1083 0.1083 0.1083
0.1083 0.1083 0.1083 0.1083 0 -0.2968 -0.0318
8.3. Completing a Two-Mode Ultrametric to one Defined on $S_A \cup S_B$

| 0.1083 | 0.1083 | 0.1083 | 0.1083 | -0.2968 | 0 | -0.0318 |
| 0.1083 | 0.1083 | 0.1083 | 0.1083 | -0.0318 | -0.0318 | 0 |
| 0.1083 | 0.1083 | 0.1083 | 0.1083 | -0.0318 | -0.0318 | -0.1516 |
| 0.1083 | 0.1083 | 0.1083 | 0.1083 | -0.0318 | -0.0318 | -0.1516 |
| 0.1083 | 0.1083 | 0.1083 | 0.1083 | 0.0520 | 0.0520 | 0.0520 |

Columns 8 through 10

| 0.1083 | 0.1083 | 0.1083 |
| 0.1083 | 0.1083 | 0.1083 |
| 0.1083 | 0.1083 | 0.1083 |
| 0.1083 | 0.1083 | 0.1083 |
| -0.0318 | -0.0318 | 0.0520 |
| -0.0318 | -0.0318 | 0.0520 |
| -0.1516 | -0.1516 | 0.0520 |
| 0 | -0.2099 | 0.0520 |
| -0.2099 | 0 | 0.0520 |
| 0.0520 | 0.0520 | 0 |

orderperm =

```
7  9  5  3  10  4  2  6  8  1
```

8.3.1 The goldfish_receptor data

We could also illustrate the results of using our various m-files from this chapter on
the two-mode goldfish_receptor data, but given the extensiveness of the output,
we just give the commands we would use and have the reader so provide the output.
The VAF value for the best ultrametric found was .6209; the best additive tree had
VAF .8663. As to be expected, the various colors are associated with the appropriate
cones.

```
>> load goldfish_receptor.dat
>> [find,vaf] = ultrafndtm(goldfish_receptor,randperm(11),randperm(9));
```

vaf =

```
0.6209
```

```
>> [ultracomp] = ultracomptm(find);
>> [orderprox,orderperm] = ultraorder(ultracomp);

>> [find,vaf,ultra,lengths] = ...
atreefndtm(goldfish_receptor,randperm(11),randperm(9));
```
vaf =

0.8663

>> [ultracompp] = ultracomptm(ultra);
Part III

The Representation of Proximity Matrices by Structures Dependent on Order (Only)
An Introduction to Order-Theoretic Representational Structures

Nonmetric multidimensional scaling (NMDS) as developed by Shepard (1962a,b) and Kruskal (1964a,b), has become a very familiar method in the psychological research literature for representing structure that may be inherent among a set of objects. Judging by the number of published substantive applications, whenever data are given in the form of a symmetric proximity matrix containing numerical relationship information between distinct object pairs, NMDS may have now become the default method of analysis. This routine use of NMDS, however, when faced with elucidating whatever pattern of relationships may underly a given set of proximities, does have interpretive implications and consequences. For one, there is an implicit choice made that whatever major generality will be allowed should reside in the particular proximities being fitted by the explicitly parameterized (Euclidean) spatial structure. Thus, an optimal (usually monotonic) transformation of the proximities is sought in conjunction with the construction of a spatial representation. Second, the parameterized spatial structure implicitly involves fitting the (transformed) proximities by some function of the differences in object placement along a set of coordinate axes that may be best suited for representing object variation that could, at least in theory, be allowed to vary continuously. For instance, in the common Euclidean model we use the square root of the sum of squared coordinate differences along a set of axes (although the particular axis system selected is open to some arbitrariness). The tacit implication is that if the structure underlying the proximities is more classificatory (and discrete) in nature, we may not do well in representing it by a spatial model that should do much better in the presence of more continuous variation (cf. Pruzansky, Tversky, and Carroll, 1982). In fact, in the limiting case where there exists a partition of the object set in which all proximities for object pairs within an object class are smaller than for object pairs between classes (and where proximities are keyed as dissimilarities so that larger values represent more dissimilar objects), NMDS will typically give a degenerate representation in which all objects within each class are located at the same spatial location and the optimally transformed proximities consist of just two values,
one for the within-class proximities and one for the between-class proximities (cf. Shepard, 1974).

This part of the book concentrates on an alternative approach to understanding what a given proximity matrix may be depicting about the objects on which it was constructed, and one that does not require a prior commitment to the sole use of either some form of dimensional model (as in NMDS), or one that is strictly classificatory (as in the use of a partition hierarchy and the implicit fitting of an ultrametric that serves as the representational mechanism for the hierarchical clustering). The method of analysis is based on approximating a given proximity matrix additively by a sum of matrices, where each component in the sum is subject to specific patterning restrictions on its entries. The restrictions imposed on each component of the decomposition (to be referred to as matrices with anti-Robinson forms) are very general and encompass interpretations that might be dimensional, or classificatory, or some combination of both (e.g., through object classes that are themselves placed dimensionally in some space). Thus, as one special case — and particularly when an (optimal) transformation of the proximities is also permitted (as we will generally allow), proximity matrices that are well interpretable through NMDS should also be interpretable through an additive decomposition of the (transformed) proximity matrix. Alternatively, when classificatory structures of various kinds might underlie a set of proximities (and the direct use of NMDS could possibly lead to a degeneracy), additive decompositions may still provide an analysis strategy for elucidating the structure.

The algorithmic details of fitting to a given proximity matrix a sum of matrices each having the desired general patterning to its entries (or even more explicitly parameterized forms that may be of help in providing a detailed interpretation, such as those given by partition hierarchies or unidimensional scales), are available in a series of papers (i.e., Hubert and Arabie, 1994, 1995a, 1995b; Hubert, Arabie, and Meulman, 1997, 1998). Thus, in this sequel we can merely refer to these sources for the actual mechanics of carrying out the various decompositions. More unique aspects that will be incorporated in the documentation to follow are (a) the possible integration of (optimal) transformations for use with the originally given proximities to be fitted by an additive matrix decomposition, and (b) the fitting of more restrictive parameterized forms (such as in Parts I and II) to the various components of a decomposition in attempting to give a detailed substantive interpretation of what each separate matrix in the decomposition may be depicting. In this latter instance, one of our concerns might be directed toward the issue of whether a particular matrix as part of a decomposition is indicating primarily dimensional or classificatory aspects of the original proximities (or possibly and what may be more typical, some combination of the two). In these latter cases, the m-files discussed as part of the documentation given in the earlier parts of this monograph for linear unidimensional scaling and for the fitting of tree structures are particularly relevant.
Chapter 9

Anti-Robinson (AR) Matrices for Symmetric Proximity Data

Denoting an arbitrary symmetric \( n \times n \) matrix by \( A = \{a_{ij}\} \), where the main diagonal entries are considered irrelevant and assumed to be zero (i.e., \( a_{ii} = 0 \) for \( 1 \leq i \leq n \)), \( A \) is said to have an anti-Robinson (AR) form if after some reordering of the rows and columns of \( A \), the entries within each row and column have a distinctive pattern: moving away from the zero main diagonal entry within any row or any column, the entries never decrease. Generally, matrices having AR forms can appear both in spatial representations for a set of proximities as functions of the absolute differences in coordinate values along some axis, or for classificatory structures that are characterized through an ultrametric.

To illustrate, we first let \( P = \{p_{ij}\} \) be a given \( n \times n \) proximity (dissimilarity) matrix among the distinct pairs of \( n \) objects in a set \( S = \{O_1, O_2, \ldots, O_n\} \) (where \( p_{ii} = 0 \) for \( 1 \leq i \leq n \)). Then, suppose, for example, a two-dimensional Euclidean representation is possible for \( P \) and its entries are very well representable by the distances in this space, so

\[
p_{ij} \approx \sqrt{(x_{1i} - x_{1j})^2 + (x_{2i} - x_{2j})^2},
\]

where \( x_{ki} \) and \( x_{kj} \) are the coordinates on the \( k^{th} \) axis (for \( k = 1 \) and 2) for objects \( O_i \) and \( O_j \) (and the symbol \( \approx \) is used to indicate approximation). Here, a simple monotonic transformation (squaring) of the proximities should then be fitted well by the sum of two matrices both having AR forms, i.e.,

\[
\{p_{ij}^2\} \approx \{(x_{1i} - x_{1j})^2\} + \{(x_{2i} - x_{2j})^2\}.
\]

In a classificatory framework, if \( \{p_{ij}\} \) were well representable, say, as a sum of two matrices, \( A_1 = \{a_{ij}^{(1)}\} \) and \( A_2 = \{a_{ij}^{(2)}\} \), each satisfying the ultrametric inequality, i.e., \( a_{ij}^{(k)} \leq \max\{a_{ih}^{(k)}, a_{hj}^{(k)}\} \) for \( k = 1 \) and 2, then

\[
\{p_{ij}\} \approx \{a_{ij}^{(1)}\} + \{a_{ij}^{(2)}\},
\]

and each of the constituent matrices can be reordered to display an AR form. As can be seen in Part II of this manual, any matrix whose entries satisfy the ultrametric
inequality can be represented by a sequence of partitions that are hierarchically related.

Given some proximity matrix $P$, the task of approximating it as a sum of matrices each having an AR form is implemented through an iterative optimization strategy based on a least-squares loss criterion that is discussed in detail by Hubert and Arabie (1994). Given the manner in which the optimization process is carried out sequentially, each successive AR matrix in any decomposition generally accounts for less and less of the patterning of the original proximity information (and very analogous to what is typically observed in a principal component decomposition of a covariance matrix). In fact, it has been found empirically that for the many data sets we have analyzed, only a very small number of such AR matrices are ever necessary to represent almost all of the patterning in the given proximities. As a succinct summary that we could give to this empirical experience: no more than three AR matrices are ever necessary; two are usually sufficient; and sometimes one will suffice.

The substantive challenge that remains, once a well-fitting decomposition is found for a given proximity matrix, is to interpret substantively what each term in the decomposition might be depicting. The strategy that could be followed would approximate each separate AR matrix by ones having a more restrictive form, and usually those representing some type of unidimensional scale (a dimensional interpretation from Part I) or partition hierarchy (a classificatory interpretation from Part II).

### 9.0.2 Incorporating Transformations

One generalization that we will now allow to what has already been discussed in the literature for fitting sums of AR matrices to a proximity matrix $P$, is the possible inclusion of an (optimal) transformation of the proximities. Thus, instead of just representing $P$ as a sum of $K$ matrices (and generally, for $K$ very small) that we might denote as $A_1 + \cdots + A_K$, where each $A_k$, $1 \leq k \leq K$, has an AR form, an (optimally) transformed matrix $\tilde{P} = \{\tilde{p}_{ij}\}$ will be fitted by such a sum, say, $\tilde{A}_1 + \cdots + \tilde{A}_K$, where the entries in $\tilde{P}$ are monotonic with respect to those in $P$, i.e., for all $O_i, O_j, O_k, O_l \in S$, $p_{ij} < p_{kl} \Rightarrow \tilde{p}_{ij} \leq \tilde{p}_{kl}$. In the sequel we will rely on the m-file, `proxmon.m`, documented in Part I, which constructs optimal monotonic transformations by the same method of isotonic regression commonly used in NMDS (although using a different type of algorithm based on the Dykstra-Kaczmarz iterative projection strategy. The method is Kruskal’s (1964a,b]) primary approach to tied proximities in $P$ that are allowed to be untied after transformation). Such transformations, for example, form the default option in the implementation of NMDS in the program KYST-2A (Kruskal, Young, and Seery, 1977) and in SYSTAT (Wilkinson, 1988).

The process of finding $\tilde{P}$ and $\tilde{A}_1 + \cdots + \tilde{A}_K$ proceeds iteratively, with the original proximity matrix $P$ first fit by $A_1 + \cdots + A_K$; a subsequent optimal (monotonic) transformation of $P$ (through a least-squares approximation to $A_1 + \cdots + A_K$) is identified, which is then refitted by the matrix sum. In many cases, this whole process can now be cycled through iteratively until convergence, i.e., through a
sequential fitting and refitting of the optimally transformed proximities and its representation as a sum of matrices each having an AR form.

In some contexts, however (particularly when fitting a single AR matrix [i.e., when \( K = 1 \)], it is probably best not to proceed to a complete convergence but instead to terminate the process after only a single optimal monotonic transformation of \( P \) is identified and then to refit by a matrix sum. This usage will be referred to as a single iteration optimal transformation (SIOT). If carried through to convergence, a perfect representation may be obtained but only at the expense of losing almost all the patterning contained within the original proximity matrix. For example, in fitting a single AR matrix, the optimal transformation identified after convergence might consist of just two values, with one corresponding to the smallest proximity in the original matrix and all others equal. Although technically permissible since this situation does reflect a perfect AR form, most of the detail present in the original proximity matrix is also lost. Difficulties with such so-called degeneracies have been pointed out by Carroll (1992), particularly when faced with fitting classificatory structures to a given proximity matrix.

9.0.3 Interpreting the Structure of an AR matrix

In representing a proximity matrix \( P \) as a sum, \( A_1 + \cdots + A_K \) (or an optimal transformation \( \tilde{P} \) as \( \tilde{A}_1 + \cdots + \tilde{A}_K \)), the interpretive task remains to explain substantively what each term of the decomposition might be depicting. We suggest four possible strategies below, with the first two attempting to understand the structure of an AR matrix directly and without much loss of detail; the last two require the imposition of strictly parameterized approximations in the form of either an ultrametric or a unidimensional scale. In the discussion below, \( A = \{a_{ij}\} \) will be assumed to have an AR form that is displayed by the given row and column order.

(A) Complete representation and reconstruction through a collection of subsets and associated subset diameters:

The entries in any AR matrix \( A \) can be reconstructed exactly through a collection of \( M \) subsets of the original object set \( S = \{O_1, \ldots, O_n\} \), denoted by \( S_1, \ldots, S_M \), and where \( M \) is determined by the particular pattern of tied entries, if any, in \( A \). These \( M \) subsets have the following characteristics:

(i) each \( S_m, 1 \leq m \leq M \), consists of a sequence of (two or more) consecutive integers so that \( M \leq n(n-1)/2 \). (This bound holds because the number of different subsets having consecutive integers for any given fixed ordering is \( n(n-1)/2 \), and will be achieved if all the entries in the AR matrix \( A \) are distinct).

(ii) each \( S_m, 1 \leq m \leq M \), has a diameter, denoted by \( d(S_m) \), so that for all object pairs within \( S_m \), the corresponding entries in \( A \) are less than or equal to the diameter. The subsets, \( S_1, \ldots, S_M \), can be assumed ordered as \( d(S_1) \leq d(S_2) \leq \cdots \leq d(S_M) \), and if \( S_m \subseteq S_{m'} \), \( d(S_m) \leq d(S_{m'}) \).

(iii) each entry in \( A \) can be reconstructed from \( d(S_1), \ldots, d(S_M) \), i.e., for \( 1 \leq i, j \leq n \),

\[
a_{ij} = \min_{1 \leq m \leq M} \{d(S_m) \mid O_i, O_j \in S_m\},
\]
so that the minimum diameter for subsets containing an object pair \( O_i, O_j \in S \) is equal to \( a_{ij} \). Given \( A \), the collection of subsets \( S_1, \ldots, S_M \) and their diameters can be identified by inspection through the use of an increasing threshold that starts from the smallest entry in \( A \), and observing which subsets containing contiguous objects emerge from this process. The substantive interpretation of what \( A \) is depicting reduces to explaining why those subsets with the smallest diameters are so homogenous. For convenience of reference, the subsets \( S_1, \ldots, S_M \) will be referred to as the set of AR reconstructive subsets.

(B) Representation by a strongly anti-Robinson matrix:

If the matrix \( A \) has a somewhat more restrictive form than just being AR, and is also strongly anti-Robinson (SAR), a convenient graphical representation can be given to the collection of AR reconstructive subsets \( S_1, \ldots, S_M \) and their diameters, and how they can serve to retrieve \( A \). Specifically, \( A \) is said to be strongly anti-Robinson (SAR) if (considering the above-diagonal entries of \( A \)) whenever two entries in adjacent columns are equal \( (a_{ij} = a_{(i+1)j}) \), those in the same two adjacent columns in the previous row are also equal \( (a_{(i-1)j} = a_{(i-1)(j+1)}) \) for \( 1 \leq i-1 < j \leq n-1 \); also, whenever two entries in adjacent rows are equal \( (a_{ij} = a_{(i+1)j}) \), those in the same two adjacent rows in the succeeding column are also equal \( (a_{ij} = a_{(i+1)(j+1)}) \) for \( 2 \leq i+1 < j \leq n-1 \).

When \( A \) is SAR, the collection of subsets, \( S_1, \ldots, S_M \), and their diameters, and how these serve to reconstruct \( A \) can be modeled graphically as we will see in Section 9.5. The internal nodes (represented by solid circles) in each of these figures are at a height equal to the diameter of the respective subset; the consecutive objects forming that subset are identifiable by downward paths from the internal nodes to the terminal nodes corresponding to the objects in \( S = \{O_1, \ldots, O_n\} \) (represented by labeled open circles). An entry \( a_{ij} \) in \( A \) can be reconstructed as the minimum node height of a subset for which a path can be constructed from \( O_i \) up to that internal node and then back down to \( O_j \). (To prevent undue graphical “clutter”, only the most homogenous subsets from \( S_1, \ldots, S_M \) having the smallest diameters should actually be included in the graphical representation of an SAR matrix; each figure would explicitly show only how the smallest entries in \( A \) can be reconstructed, although each could be easily extended to include all of \( A \). The calibrated vertical axis in such figures could routinely include the heights at which the additional internal nodes would have to be placed to effect such a complete reconstruction.)

Given an arbitrary AR matrix \( A \), a least-squares SAR approximating matrix to \( A \) can be found using the heuristic optimization search strategy illustrated in Section 9.3 and developed in Hubert, Arabie, and Meulman (1998). This latter source also discusses in detail (through counterexample) why strongly AR conditions need to be imposed to obtain a consistent graphical representation.

(C) Representation by a unidimensional scale:

To obtain greater graphical simplicity for an eventual substantive interpretation than offered by an SAR matrix, one possibility is to use approximating unidimensional scales. To be explicit, one very simple form that an AR matrix \( A \) may assume is interpretable by a single dimension and through a unidimensional scale
in which the entries have the parameterized form, \( A = \{a_{ij}\} = \{|x_j - x_i| + c\} \), where the coordinates are ordered as \( x_1 \leq x_2 \leq \cdots \leq x_n \) and \( c \) is an estimated constant. Given any proximity matrix, a least-squares approximating unidimensional scale can be obtained through the optimization strategies of Part I, and would be one (dimensional) method that could be followed in attempting to interpret what a particular AR component of a decomposition might be revealing.

(D) Representation by an ultrametric:

A second simple form that an AR matrix \( A \) could have is strictly classificatory in which the entries in \( A \) satisfy the ultrametric condition: \( a_{ij} \leq \max\{a_{ik}, a_{jk}\} \) for all \( O_i, O_j, O_k \in S \). As a threshold is increased from the smallest entry in \( A \), a sequence of partitions of \( S \) is identified in which each partition is constructed from the previous one by uniting pairs of subsets from the latter. A partition identified at a given threshold level has equal values in \( A \) between each given pair of subsets, and all the within subset values are not greater than the between subset values. The reconstructive subsets \( S_1, \ldots, S_M \) that would represent the AR matrix \( A \) are now the new subsets that are formed in the sequence of partitions, and have the property that if \( d(S_m) \leq d(S_m') \), then \( S_m \subseteq S_m' \) or \( S_m \cap S_m' = \emptyset \). Given any proximity matrix, a least-squares approximating ultrametric can be constructed by the heuristic optimization routines developed in Part II, and would be another (classificatory) strategy for interpreting what a particular AR component of a decomposition might be depicting. As might be noted, there are generally \( n - 1 \) subsets (each of size greater than one) in the collection of reconstructive subsets for any ultrametric, and thus \( n - 1 \) values need to be estimated in finding the least-squares approximation (which is the same number needed for a least-squares approximating unidimensional scale, based on obtaining the \( n - 1 \) non-negative separation values between \( x_i \) and \( x_{i+1} \) for \( 1 \leq i \leq n - 1 \)).

9.1 Fitting a Given AR Matrix in the \( L_2 \)-Norm

The function m-file, \texttt{arobfit.m}, fits an anti-Robinson matrix using iterative projection to a symmetric proximity matrix in the \( L_2 \)-norm. The usage syntax is of the form

\[
[\text{fit}, \text{vaf}] = \text{arobfit} (\text{prox}, \text{inperm})
\]

where \( \text{PROX} \) is the input proximity matrix \( (n \times n \text{ with a zero main diagonal and a dissimilarity interpretation}) \); \( \text{INPERM} \) is a given permutation of the first \( n \) integers; \( \text{FIT} \) is the least-squares optimal matrix (with variance-accounted-for of \( \text{VAF} \)) to \( \text{PROX} \) having an anti-Robinson form for the row and column object ordering given by \( \text{INPERM} \). A recording of a MATLAB session using the \texttt{number.dat} data file and object ordering given by the identity permutation follows:

\[
\text{load number.dat}
\text{inperm} = 1:10
\text{inperm} =
\]
123456789 1 0
[fit, vaf] = arobfit(number, inperm)

fit =

Columns 1 through 7

<p>| | | | | | | |</p>
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<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.5840</td>
<td>0.6965</td>
<td>0.6965</td>
<td>0.7960</td>
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vaf =

0.6979

9.1.1 Fitting the (In)-equality Constraints Implied by a Given Matrix in the $L_2$-Norm

At times it may be useful to fit through iterative projection a given set of equality and inequality constraints (as represented by the equalities and inequalities present among the entries in a given target matrix) to a symmetric proximity matrix in the $L_2$-norm. Whenever the target matrix is AR in form already, the resulting fitted matrix would also be AR in form; more generally, however, the m-function,
9.1. Fitting a Given AR Matrix in the $L_2$-Norm

`targfit.m` could be used with any chosen target matrix. The usage follows the form

```
[fit,vaf] = targfit(prox,targ)
```

where, as usual, `PROX` is the input proximity matrix (with a zero main diagonal and a dissimilarity interpretation); `TARG` is a matrix of the same size as `PROX`; `FIT` is the least-squares optimal matrix (with variance-accounted-for of `VAF`) to `PROX` satisfying the equality and inequality constraints implicit among all the entries in `TARG`. An example follows in which the given target matrix is a distance matrix (having an AR form) between equally-spaced object placements along a line; the resulting fitted matrix obviously has an AR form as well:

```
load number.dat
[fit,vaf] = targfit(number,targlin(10))
```

```
fit =

Columns 1 through 7

0   0.3714  0.3714  0.5363  0.5363  0.6548  0.6548
0.3714    0   0.3714  0.3714  0.5363  0.5363  0.6548
0.3714  0.3714      0   0.3714  0.3714  0.5363  0.5363
0.5363  0.3714  0.3714      0   0.3714  0.3714  0.5363
0.5363  0.5363  0.3714  0.3714      0   0.3714  0.3714
0.6548  0.5363  0.5363  0.3714  0.3714      0   0.3714
0.6548  0.6548  0.5363  0.5363  0.3714  0.3714      0
0.7908  0.7908  0.6548  0.6548  0.5363  0.5363  0.3714
0.7908  0.7908  0.6548  0.6548  0.5363  0.5363  0.3714
0.8500  0.7908  0.7908  0.6548  0.6548  0.5363  0.5363

Columns 8 through 10

0.7908  0.7908  0.8500
0.6548  0.7908  0.7908
0.6548  0.6548  0.7908
0.5363  0.6548  0.6548
0.5363  0.5363  0.6548
0.3714  0.5363  0.5363
0.3714  0.3714  0.5363
0.3714      0   0.3714
0.3714  0.3714      0

vaf =

0.5105
```
9.2 Finding an AR Matrix in the $L_2$-Norm

The fitting of a given AR matrix by the m-function of Section 9.1, \texttt{arobfit.m}, requires the presence of an initial permutation to direct the optimization process. Thus, the finding of a best-fitting AR matrix reduces to the identification of an appropriate object permutation to use \textit{ab initio}. We suggest the adoption of \texttt{order.m}, which carries out an iterative Quadratic Assignment (QA) maximization task using a given square ($n \times n$; one-way, one-way) proximity matrix \texttt{PROX} (with a zero main diagonal and a dissimilarity interpretation) (see Section 1.2.1 and the references given there to the literature of QA). Three separate local operations are used to permute the rows and columns of the proximity matrix to maximize the cross-product index with respect to a given square target matrix \texttt{TARG}: (a) pairwise interchanges of objects in the permutation defining the row and column order of the square proximity matrix; (b) the insertion of from 1 to \texttt{KBLOCK} (which is less than or equal to $n - 1$) consecutive objects in the permutation defining the row and column order of the data matrix; and (c) the rotation of from 2 to \texttt{KBLOCK} (which is less than or equal to $n - 1$) consecutive objects in the permutation defining the row and column order of the data matrix. The usage syntax has the form

\[ \texttt{[outperm,rawindex,allperms,index] = order(prox,targ,inperm,kblock)} \]

where \texttt{INPERM} is the input beginning permutation (a permutation of the first $n$ integers); \texttt{OUTPERM} is the final permutation of \texttt{PROX} with the cross-product index \texttt{RAWINDEX} with respect to \texttt{TARG}. The cell array \texttt{ALLPERMS} contains \texttt{INDEX} entries corresponding to all the permutations identified in the optimization from \texttt{ALLPERMS(1)} = \texttt{INPERM} to \texttt{ALLPERMS(INDEX)} = \texttt{OUTPERM}.

A recording of a MATLAB session using \texttt{order.m} is listed below with the beginning \texttt{INPERM} given as the identity permutation, \texttt{TARG} by an equally-spaced object placement along a line, and \texttt{KBLOCK} = 3. Using the generated \texttt{OUTPERM}, \texttt{arobfit.m} is then invoked to fit an AR form having final VAF of .7782.

\begin{verbatim}
load number.dat
targlinear = targlin(10);
[outperm,rawindex,allperms,index] = order(number,targlinear,1:10,3)
outperm =
    1  2  3  5  4  6  7  9  10  8
rawindex =
  206.4920
allperms =
    [1x10 double]    [1x10 double]    [1x10 double]    [1x10 double]
\end{verbatim}
9.2. Finding an AR Matrix in the $L_2$-Norm

index =

4

[fit, vaf] = arobfit(number, outperm)

fit =

Columns 1 through 7

\begin{array}{cccccccc}
 0 & 0.4210 & 0.5840 & 0.6840 & 0.7090 & 0.7960 & 0.7960 \\
 0.4210 & 0 & 0.2840 & 0.4960 & 0.4960 & 0.5880 & 0.7357 \\
 0.5840 & 0.2840 & 0 & 0.0590 & 0.3835 & 0.4928 & 0.4928 \\
 0.6840 & 0.4960 & 0.0590 & 0 & 0.3835 & 0.3985 & 0.3985 \\
 0.7090 & 0.4960 & 0.3835 & 0.3835 & 0 & 0.3750 & 0.3750 \\
 0.7960 & 0.5880 & 0.3985 & 0.3985 & 0.3750 & 0 & 0.3750 \\
 0.7960 & 0.7357 & 0.4928 & 0.3985 & 0.3750 & 0.3750 & 0 \\
 0.8210 & 0.7357 & 0.4928 & 0.4928 & 0.4928 & 0.4928 & 0.3460 \\
 0.8500 & 0.7357 & 0.7357 & 0.6830 & 0.4928 & 0.4928 & 0.3460 \\
 0.9090 & 0.7357 & 0.7357 & 0.7357 & 0.5920 & 0.4928 & 0.4253
\end{array}

Columns 8 through 10

\begin{array}{cccc}
 0.8210 & 0.8500 & 0.9090 \\
 0.7357 & 0.7357 & 0.7357 \\
 0.4928 & 0.7357 & 0.7357 \\
 0.4928 & 0.6830 & 0.7357 \\
 0.4928 & 0.4928 & 0.5920 \\
 0.4928 & 0.4928 & 0.4928 \\
 0.3460 & 0.3460 & 0.4253 \\
 0 & 0.3460 & 0.4253 \\
 0.3460 & 0 & 0.4253 \\
 0.4253 & 0.4253 & 0
\end{array}

vaf =

0.7782

The m-file, arobfnd.m is our preferred method for actually identifying a single AR form, and incorporates an initial equally-spaced target and uses the iterative QA routine of order.m to generate better permutations; the obtained AR forms then are used as new targets against which possibly even better permutations might be identified, until convergence (i.e., the identified permutations remain the same).
The syntax is as follows:

```
[fit, vaf, outperm] = arobfnd(prox, inperm, kblock)
```

where \( \text{PROX} \) is the input proximity matrix (\( n \times n \) with a zero main diagonal and a dissimilarity interpretation); \( \text{INPERM} \) is a given starting permutation of the first \( n \) integers; \( \text{FIT} \) is the least-squares optimal matrix (with variance-accounted-for of \( \text{VAF} \)) to \( \text{PROX} \) having an anti-Robinson form for the row and column object ordering given by the ending permutation \( \text{OUTPERM} \); \( \text{KBLOCK} \) defines the block size in the use the iterative quadratic assignment routine.

As seen from the example below, and starting from a random initial permutation, the same AR form is found as with just one application of \texttt{order.m} reported above.

```
[fit, vaf, outperm] = arobfnd(number, randperm(10), 1);
```

```
vaf =

0.7782
```

```
outperm =

8    10     9     7     6     4     5     3     2     1
```

### 9.3 Fitting and Finding a Strongly Anti-Robinson (SAR) Matrix in the \( L_2 \)-Norm

The \textit{m}-functions, \texttt{sarobfit.m} and \texttt{sarobfnd.m}, are direct analogues of \texttt{arobfit.m} and \texttt{arobfnd.m}, respectively, but are concerned with fitting and finding \textit{strongly} anti-Robinson forms. The syntax for \texttt{sarobfit.m}, which fits a strongly anti-Robinson matrix using iterative projection to a symmetric proximity matrix in the \( L_2 \)-norm, is

```
[fit, vaf] = sarobfit(prox, inperm)
```

where, again, \( \text{PROX} \) is the input proximity matrix (\( n \times n \) with a zero main diagonal and a dissimilarity interpretation); \( \text{INPERM} \) is a given permutation of the first \( n \) integers; \( \text{FIT} \) is the least-squares optimal matrix (with variance-accounted-for of \( \text{VAF} \)) to \( \text{PROX} \) having a strongly anti-Robinson form for the row and column object ordering given by \( \text{INPERM} \).

An example follows using the same identity permutation as was implemented in fitting an AR form with \texttt{arobfit.m}; as might be expected from using the more restrictive strongly anti-Robinson form, the \textit{VAF} drops to .6128 from .6979.

```
load number.dat
[fit, vaf] = sarobfit(number,1:10)
```
### 9.3. Fitting and Finding a Strongly Anti-Robinson (SAR) Matrix in the \( L_2 \)-Norm 127

**fit =**

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</tbody>
</table>

**vaf =**

0.6128

The m-function, `sarobfnd.m`, fits a strongly anti-Robinson matrix using iterative projection to a symmetric proximity matrix in the \( L_2 \)-norm based on a permutation identified through the use of iterative quadratic assignment. The function has the expected syntax

\[
[\text{fit}, \text{vaf}, \text{outperm}] = \text{sarobfnd}(\text{prox}, \text{inperm}, \text{kblock})
\]

where, again, `PROX` is the input proximity matrix (\( n \times n \) with a zero main diagonal and a dissimilarity interpretation); `INPERM` is a given starting permutation of the first \( n \) integers; `FIT` is the least-squares optimal matrix (with variance-accounted-for of `VAF`) to `PROX` having a strongly anti-Robinson form for the row and column object ordering given by the ending permutation `OUTPERM`. As usual, `KBLOCK` defines the block size in the use the iterative quadratic assignment routine.

In the MATLAB recording below, and starting from a random permutation, a strongly anti-Robinson form is found with a `VAF` of .7210 (an expected drop from...
the value of .7782 for the anti-Robinson form found using `arobfnd.m`.

```
[find,vaf,outperm] = sarobfnd(number,randperm(10),1)

find =

Columns 1 through 7

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<td>0.7227</td>
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Columns 8 through 10

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vaf =

0.7210

outperm =

1 2 3 5 4 6 7 10 9 8
9.4 The Use of Optimal Transformations and the m-function proxmon.m

As previously discussed in Part I, the function \texttt{proxmon.m}, provides a monotonically transformed proximity matrix that is close in a least-squares sense to a given input matrix. The syntax is

\[
[\text{monproxpermut vaf diff}] = \text{proxmon(\text{proxpermut}, \text{fitted})}
\]

where \texttt{PROXPERMUT} is the input proximity matrix (which may have been subjected to an initial row/column permutation, hence the suffix 'PERMUT'); and \texttt{FITTED} is a given target matrix; the output matrix \texttt{MONPROXPERMUT} is closest to \texttt{FITTED} in a least-squares sense and obeys the order constraints obtained from each pair of entries in (the upper-triangular portion of) \texttt{PROXPERMUT} (and where the inequality constrained optimization is carried out using the Dykstra-Kaczmarz iterative projection strategy); \texttt{VAF} indicates how much variance in \texttt{MONPROXPERMUT} can be accounted for by \texttt{FITTED}; finally, \texttt{DIFF} is the value of the least-squares loss function and is (one-half) the sum of squared differences between the entries in \texttt{MONPROXPERMUT} and \texttt{FITTED}.

In the notation of the introduction when fitting a given order, \texttt{FITTED} would correspond to the AR matrix \(A = \{a_{ij}\}\); the input \texttt{PROXPERMUT} would be \(\{p_{\rho(i)\rho(j)}\}\); \texttt{MONPROXPERMUT} would be \(\{f(p_{\rho(i)\rho(j)})\}\), where the function \(f(\cdot)\) satisfies the monotonicity constraints, i.e., if \(p_{\rho(i)\rho(j)} < p_{\rho(i')\rho(j')}\) for \(1 \leq i < j \leq n\) and \(1 \leq i' < j' \leq n\), then \(f(p_{\rho(i)\rho(j)}) \leq f(p_{\rho(i')\rho(j')})\). The transformed proximity matrix \(\{f(p_{\rho(i)\rho(j)})\}\) minimizes the least-squares criterion (\texttt{DIFF}) of

\[
\sum_{i<j}(f(p_{\rho(i)\rho(j)}) - a_{ij})^2,
\]

over all functions \(f(\cdot)\) that satisfy the monotonicity constraints. The \texttt{VAF} is a normalization of this loss value by the sum of squared deviations of the transformed proximities from their mean:

\[
\text{VAF} = 1 - \frac{\sum_{i<j}(f(p_{\rho(i)\rho(j)}) - a_{ij})^2}{\sum_{i<j}(f(p_{\rho(i)\rho(j)}) - \bar{f})^2},
\]

where \(\bar{f}\) denotes the mean of the off-diagonal entries in \(\{f(p_{\rho(i)\rho(j)})\}\).

The script m-file listed below gives an application of \texttt{proxmon.m} along with finding a best-fitting AR form for our \texttt{number.dat} matrix. First, \texttt{arobfnd.m} is invoked to obtain a best-fitting AR matrix (\texttt{fit}); this is the same as found earlier based on the \texttt{outperm} of \([1\ 2\ 3\ 5\ 4\ 6\ 7\ 9\ 10\ 8]\) and generating a \texttt{VAF} of .7782. The m-file, \texttt{proxmon.m}, is then used to generate the monotonically transformed proximity matrix (\texttt{monproxpermut}) with \texttt{VAF} of .8323. Given the SIOT (single-iteration-optimal-transformation) discussed in the introduction, it might now be best to fit once more an AR matrix to this now monotonically transformed proximity matrix, but then stop. Otherwise, as seen in the output below, if the strategy is repeated cyclically (i.e., finding a fitted matrix based on the monotonically transformed proximity matrix, finding a new monotonically transformed matrix, and so
on), a perfect VAF of 1.0 can be achieved at the expense of losing most of the detail in the transformed proximities, i.e., only five distinct values remain that correspond to the three largest and single smallest of the original proximities with all the remaining now tied at a value of .5467. (To avoid another type of degeneracy [where all matrices would converge to zeros], the sum of squares of the fitted matrix was maintained stationary; convergence is based on observing a minimal change (less than 1.0e-010) in the VAF.

load number.dat
[fit vaf outperm] = arobfnd(number,randperm(10),2)
[monproxpermut vaf diff] = ...
    proxmon(number(outperm, outperm), fit)
sumfitsq = sum(sum(fit.^2));
prevvaf = 2;
while (abs(prevvaf-vaf) >= 1.0e-010)
    prevvaf = vaf;
    [fit vaf] = arobfit(monproxpermut,1:10);
    sumnewfitsq = sum(sum(fit.^2));
    fit = sqrt(sumfitsq)*(fit/sqrt(sumnewfitsq));
    [monproxpermut vaf diff] = proxmon(number(outperm, outperm), fit);
end

outperm
fit

monproxpermut
vaf
diff

fit =

Columns 1 through 7

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</table>

Columns 8 through 10
9.4. The Use of Optimal Transformations and the m-function proxmon.m

\[
\begin{array}{ccc}
0.8210 & 0.8500 & 0.9090 \\
0.7357 & 0.7357 & 0.7357 \\
0.4928 & 0.7357 & 0.7357 \\
0.4928 & 0.6830 & 0.7357 \\
0.4928 & 0.4928 & 0.5920 \\
0.4928 & 0.4928 & 0.4928 \\
0.3460 & 0.3460 & 0.4253 \\
0 & 0.3460 & 0.4253 \\
0.3460 & 0 & 0.4253 \\
0.4253 & 0.4253 & 0
\end{array}
\]

\[\text{vaf} = \]

0.7782

\[\text{outperm} = \]

1 2 3 5 4 6 7 9 10 8

\[\text{monproxperm} = \]

**Columns 1 through 7**

\[
\begin{array}{cccccccc}
0 & 0.4244 & 0.5549 & 0.6840 & 0.7058 & 0.7659 & 0.7058 \\
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0.6840 & 0.5908 & 0.0590 & 0 & 0.4244 & 0.4244 & 0.4054 \\
0.7058 & 0.4054 & 0.4054 & 0.4244 & 0 & 0.4310 & 0.3981 \\
0.7659 & 0.5549 & 0.5908 & 0.4244 & 0.4310 & 0 & 0.4054 \\
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\]

**Columns 8 through 10**

\[
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0.3981 & 0.6830 & 0.7058 \\
0.7058 & 0.3981 & 0.5908 \\
0.5908 & 0.5549 & 0.4244
\end{array}
\]
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\[
\begin{bmatrix}
0.4054 & 0.3981 & 0.4244 \\
0 & 0.4054 & 0.4244 \\
0.4054 & 0 & 0.4310 \\
0.4244 & 0.4310 & 0
\end{bmatrix}
\]

\[
vaf = 0.8323
\]

\[
diff = 0.2075
\]

\[
outperm =
\begin{bmatrix}
1 & 2 & 3 & 5 & 4 & 6 & 7 & 9 & 10 & 8
\end{bmatrix}
\]

\[
fit =
\]

**Columns 1 through 7**

\[
\begin{bmatrix}
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0.5467 & 0.5467 & 0 & 0.0609 & 0.5467 & 0.5467 & 0.5467 \\
0.5467 & 0.5467 & 0.0609 & 0 & 0.5467 & 0.5467 & 0.5467 \\
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0.5467 & 0.5467 & 0.5467 & 0.5467 & 0.5467 & 0.5467 & 0 \\
0.8474 & 0.5467 & 0.5467 & 0.5467 & 0.5467 & 0.5467 & 0.5467 \\
0.8774 & 0.5467 & 0.5467 & 0.5467 & 0.5467 & 0.5467 & 0.5467 \\
0.9383 & 0.5467 & 0.5467 & 0.5467 & 0.5467 & 0.5467 & 0.5467
\end{bmatrix}
\]

**Columns 8 through 10**

\[
\begin{bmatrix}
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0.5467 & 0.5467 & 0.5467 \\
0.5467 & 0.5467 & 0.5467 \\
0.5467 & 0.5467 & 0.5467 \\
0.5467 & 0.5467 & 0.5467 \\
0.5467 & 0.5467 & 0.5467 \\
0 & 0.5467 & 0.5467
\end{bmatrix}
\]
9.4. The Use of Optimal Transformations and the m-function proxmon.m

\[
\begin{bmatrix}
0.5467 & 0 & 0.5467 \\
0.5467 & 0.5467 & 0 \\
\end{bmatrix}
\]

\[
\text{monproxperm} =
\]

Columns 1 through 7

\[
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0.5467 & 0.5467 & 0 & 0.0609 & 0.5467 & 0.5467 & 0.5467 \\
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0.8774 & 0.5467 & 0.5467 & 0.5467 & 0.5467 & 0.5467 & 0.5467 \\
0.9383 & 0.5467 & 0.5467 & 0.5467 & 0.5467 & 0.5467 & 0.5467 \\
\end{bmatrix}
\]

Columns 8 through 10

\[
\begin{bmatrix}
0.8474 & 0.8774 & 0.9383 \\
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0.5467 & 0.5467 & 0.5467 \\
0.5467 & 0.5467 & 0.5467 \\
0.5467 & 0.5467 & 0.5467 \\
0.5467 & 0.5467 & 0.5467 \\
0.5467 & 0.5467 & 0.5467 \\
0 & 0.5467 & 0.5467 \\
0.5467 & 0 & 0.5467 \\
0.5467 & 0.5467 & 0 \\
\end{bmatrix}
\]

\[
\text{vaf } = \ 1.0000
\]

\[
\text{diff } = \ 8.3999e-011
\]
9.5 Graphically Representing SAR Structures

The use of the very general form of representation offered by an AR matrix without the imposition of any further restrictions has one annoying interpretive difficulty. Specifically, it is usually necessary to interpret the fitted structures directly (and enumeratively) through a set of subsets or clusters that are all defined by objects contiguous in a specific object ordering; each such subset has an attached diameter that reflects its maximum within-class fitted value. More pointedly, it is generally not possible to use a more convenient graph-theoretic structure and the lengths of paths between objects in such a graph to represent visually a fitted AR matrix; this situation contrasts with opportunities resulting when the approximation matrix is more restricted and defined, say, by an ultrametric or an additive tree, or by a (linear or circular) unidimensional scaling (see Hubert, Arabie, & Meulman, 1997, or Parts I and II of the current monograph).

As noted in the introduction, the imposition of SAR conditions allows a representation of the fitted values in a (least-squares) SAR approximating matrix as lengths of paths in a graph, although this graph will not generally have the simplified form of a tree. A discussion of these latter SAR constraints is not new here, and several (theoretical) presentations of their usefulness exist in the literature (for example, see Critchley and Fichet, 1994; Critchley, 1994; Durand and Fichet, 1988; Mirkin, 1996, Chapter 7). Here, we give the example based on the number data from Hubert, Arabie, and Meulman (1998) for interpretative convenience. The latter data were transformed (in that reference) to a standard deviation of 1.0 and a mean of 4.0; thus, the numbers within the fitted matrices will differ from the examples given earlier. Approximating AR and SAR forms for the transformed number proximity data are given in the upper and lower-triangular portions, respectively, of the matrix in Table 9.1. For convenience below, we will denote the upper-triangular AR matrix by $\mathbf{A}_{ut}$ and the lower-triangular SAR matrix by $\mathbf{A}_{lt}$.

The $10(10 - 1)/2 = 45$ subsets defined by objects contiguous in the object ordering used to display the upper-triangular portion of Table 9.1 are listed in Table 9.2 according to increasing diameter values. For purposes of our later discussion, 22 of the subsets are given in italics to indicate that they are proper subsets of another listed subset having the same diameter. Substantively, the dominant patterning of the entries in $\mathbf{A}_{ut}$ appears to reflect (primarily) digit magnitude except for the placement of digit 4 next to 2, and digit 7 being located in the last position. Both these latter deviations from an interpretation strictly according to digit magnitude show some of the salient structural properties of the digits. For example, the digit pair (2,4) has the absolute smallest dissimilarity in the data; besides being relatively close in magnitude, there are the possible (although redundant) similarity bases that $2+2=4$, $2\times2=4$, 4 is a power of 2, and both 2 and 4 are even numbers. Similarly, the placement of the digit 7 in the last position results from the salience of the triple $\{6,8,9\}$, which is the third to emerge according to its diameter. In addition to these three digits all being relatively close in magnitude, 6 and 8 are both even numbers, 6 and 9 are multiples of 3, and 8 is directly adjacent in size to 9. The three original dissimilarities within the set $\{6,8,9\}$ are all smaller than the dissimilarities digit 7 has to any other digit.
Table 9.1. Order-constrained least-squares approximations to the digit proximity data of Shepard et al. (1975); the upper-triangular portion is anti-Robinson and the lower-triangular portion is strongly-anti-Robinson

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<th>digit</th>
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<th>4</th>
<th>3</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>9</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>x</td>
<td>3.41</td>
<td>4.21</td>
<td>4.70</td>
<td>4.83</td>
<td>5.25</td>
<td>5.25</td>
<td>5.38</td>
<td>5.52</td>
<td>5.81</td>
</tr>
<tr>
<td>1</td>
<td>3.41</td>
<td>x</td>
<td>2.73</td>
<td>3.78</td>
<td>3.78</td>
<td>4.23</td>
<td>4.96</td>
<td>4.96</td>
<td>4.96</td>
<td>4.96</td>
</tr>
<tr>
<td>2</td>
<td>4.21</td>
<td>2.73</td>
<td>x</td>
<td>1.63</td>
<td>3.22</td>
<td>3.76</td>
<td>3.76</td>
<td>3.76</td>
<td>4.96</td>
<td>4.96</td>
</tr>
<tr>
<td>4</td>
<td>4.76</td>
<td>3.78</td>
<td>1.63</td>
<td>x</td>
<td>3.22</td>
<td>3.30</td>
<td>3.76</td>
<td>4.70</td>
<td>4.96</td>
<td>4.96</td>
</tr>
<tr>
<td>3</td>
<td>4.76</td>
<td>3.78</td>
<td>3.22</td>
<td>3.22</td>
<td>x</td>
<td>3.18</td>
<td>3.18</td>
<td>3.76</td>
<td>3.76</td>
<td>4.25</td>
</tr>
<tr>
<td>5</td>
<td>5.25</td>
<td>4.59</td>
<td>3.53</td>
<td>3.53</td>
<td>3.18</td>
<td>x</td>
<td>3.18</td>
<td>3.76</td>
<td>3.76</td>
<td>3.76</td>
</tr>
<tr>
<td>6</td>
<td>5.25</td>
<td>4.59</td>
<td>3.53</td>
<td>3.53</td>
<td>3.18</td>
<td>3.18</td>
<td>x</td>
<td>3.04</td>
<td>3.04</td>
<td>3.43</td>
</tr>
<tr>
<td>8</td>
<td>5.57</td>
<td>4.96</td>
<td>4.18</td>
<td>4.18</td>
<td>4.18</td>
<td>4.18</td>
<td>3.04</td>
<td>x</td>
<td>3.04</td>
<td>3.43</td>
</tr>
<tr>
<td>9</td>
<td>5.57</td>
<td>4.96</td>
<td>4.18</td>
<td>4.18</td>
<td>4.18</td>
<td>4.18</td>
<td>3.04</td>
<td>x</td>
<td>3.43</td>
<td>3.43</td>
</tr>
<tr>
<td>7</td>
<td>5.57</td>
<td>4.96</td>
<td>4.18</td>
<td>4.18</td>
<td>4.18</td>
<td>4.18</td>
<td>3.43</td>
<td>3.43</td>
<td>x</td>
<td>3.43</td>
</tr>
</tbody>
</table>

Given just the collection of subsets $S_1, \ldots, S_M$ listed in Table 9.2 and their associated matrix diameters, it is possible (trivially) to reconstruct the original approximating matrix $A_{ut}$ by identifying for each object pair the smallest diameter for a subset that contains that pair. (Explicitly, the smallest diameter for a subset that contains an object pair is equal to the value in $A_{ut}$ associated with that pair, and the subset itself includes that object pair and all objects in between in the ordering that is used to display the AR form for $A_{ut}$.) This type of reconstruction is generally possible for any matrix that can be row/column reordered to an AR form through the collection of subsets $S_1, \ldots, S_M$ and their diameters identified by increasing a threshold variable from the smallest fitted value. In fact, even if all the italicized subsets were removed (that are proper subsets of another having the same diameter), exactly the same reconstruction could be carried out because the italicized subsets are redundant with respect to identifying for each object pair the smallest diameter for a subset that contains the pair.

Without imposing further restrictions on the approximating matrix other than just being AR, a more convenient representation using a graph and path lengths in such a graph is generally not possible. We will select two small (AR) submatrices from the upper-triangular portion of Table 9.1 to make this point more convincingly, and in the process indicate by example how a graph representation is to be constructed and why further restrictions on the approximating matrix may be necessary to carry out the task.

First, consider the fitted values for the first four placed digits, 0, 1, 2, and 4 in Figure 9.1(a), for which the desired type of graphical representation is possible without imposing any further constraints. This AR submatrix is given in Figure 9.1(a) along with the six corresponding subsets of contiguous objects and their diameters, and a graphical representation for the structure. The latter consists of
Table 9.2. The 45 subsets listed according to increasing diameter values that are contiguous in the object ordering used to display the upper-triangular portion of Table 9.1. The 22 subsets given in italics are redundant in the sense that they are proper subsets of another listed subset with the same diameter.

<table>
<thead>
<tr>
<th>subset</th>
<th>diameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>{2,4}</td>
<td>1.63</td>
</tr>
<tr>
<td>{1,2}</td>
<td>2.73</td>
</tr>
<tr>
<td>{6,8},{8,9},{6,8,9}</td>
<td>3.04</td>
</tr>
<tr>
<td>{3,5},{5,6},{3,5,6}</td>
<td>3.18</td>
</tr>
<tr>
<td>{4,3},{2,4,3}</td>
<td>3.22</td>
</tr>
<tr>
<td>{4,3,5},{4,3,5,6}</td>
<td>3.30</td>
</tr>
<tr>
<td>{0,1}</td>
<td>3.41</td>
</tr>
<tr>
<td>{9,7},{8,9,7},{6,8,9,7}</td>
<td>3.43</td>
</tr>
<tr>
<td>{5,6,8},{5,6,8,9},{5,6,8,9,7}</td>
<td>3.76</td>
</tr>
<tr>
<td>{3,5,6,8},{3,5,6,8,9}</td>
<td>3.76</td>
</tr>
<tr>
<td>{4,3,5,6,8},{2,4,3,5},{2,4,3,5,6},{2,4,3,5,6,8}</td>
<td>3.76</td>
</tr>
<tr>
<td>{1,2,4},{1,2,4,3}</td>
<td>3.78</td>
</tr>
<tr>
<td>{0,1,2}</td>
<td>4.21</td>
</tr>
<tr>
<td>{1,2,4,3,5}</td>
<td>4.23</td>
</tr>
<tr>
<td>{3,5,6,8,9,7}</td>
<td>4.25</td>
</tr>
<tr>
<td>{0,1,2,4},{4,3,5,6,8,9}</td>
<td>4.70</td>
</tr>
<tr>
<td>{0,1,2,4,3}</td>
<td>4.83</td>
</tr>
<tr>
<td>{1,2,4,3,5,6},{1,2,4,3,5,6,8}</td>
<td>4.96</td>
</tr>
<tr>
<td>{1,2,4,3,5,6,8,9},{2,4,3,5,6,8,9}</td>
<td>4.96</td>
</tr>
<tr>
<td>{2,4,3,5,6,8,9,7},{4,3,5,6,8,9,7}</td>
<td>4.96</td>
</tr>
<tr>
<td>{1,2,4,3,5,6,8,9,7}</td>
<td>4.96</td>
</tr>
<tr>
<td>{0,1,2,4,3,5},{0,1,2,4,3,5,6,8}</td>
<td>5.25</td>
</tr>
<tr>
<td>{0,1,2,4,3,5,6,8,9}</td>
<td>5.38</td>
</tr>
<tr>
<td>{0,1,2,4,3,5,6,8,9,7}</td>
<td>5.52</td>
</tr>
</tbody>
</table>

four nodes corresponding to the original four objects that we represent by open circles (referred to as “terminal” nodes), plus six nodes represented by solid circles that denote the six subsets in the given listing (referred to as “internal nodes”). Based on this graph and the internal node heights provided by the calibrated scale on the left, a fitted value in the submatrix between any two terminal nodes can be obtained as one-half the length of the minimum path from one of the terminal nodes up to an internal node and back down to the other terminal node. All horizontal line segments are used here for display convenience only and are not actually assumed to contribute to the length of any path. Thus, if we changed the vertical scaling by a multiplier of 1/2, each of the fitted values in the submatrix would be exactly the
9.6 Representation Through Multiple (Strongly) AR Matrices

The representation of a proximity matrix by a single anti-Robinson structure extends easily to the additive use of multiple matrices. The m-function, `biarobfnd.m`, fits the sum of two anti-Robinson matrices using iterative projection to a symmetric proximity matrix in the $L_2$-norm based on permutations identified through the use of...
Figure 9.1. Two $4 \times 4$ submatrices and the object subsets they induce, taken from the anti-Robinson matrix in the upper-triangular portion of Table 9.1. For (a), a graphical representation of the fitted values is possible; for (b), the anomaly indicated by the dashed lines prevents a consistent graphical representation from being constructed.

of iterative quadratic assignment. The usage syntax is

```
[find,vaf,targone,targtwo,outpermone,outpermtwo] = ...
biarobfnd(prox,inperm,kblock)
```
9.6. Representation Through Multiple (Strongly) AR Matrices

Table 9.3. The fourteen (nonredundant) subsets listed according to increasing diameter values are contiguous in the linear object ordering used to display the lower-triangular SAR portion of Table 9.1.

<table>
<thead>
<tr>
<th>subset</th>
<th>diameter</th>
<th>subset</th>
<th>diameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>{2,4}</td>
<td>1.63</td>
<td>{2,4,3,5,6}</td>
<td>3.53</td>
</tr>
<tr>
<td>{1,2}</td>
<td>2.73</td>
<td>{1,2,4,3}</td>
<td>3.78</td>
</tr>
<tr>
<td>{6,8,9}</td>
<td>3.04</td>
<td>{2,4,3,5,6,8,9,7}</td>
<td>4.18</td>
</tr>
<tr>
<td>{3,5,6}</td>
<td>3.18</td>
<td>{0,1,2}</td>
<td>4.21</td>
</tr>
<tr>
<td>{2,4,3}</td>
<td>3.22</td>
<td>{0,1,2,4,3}</td>
<td>4.76</td>
</tr>
<tr>
<td>{0,1}</td>
<td>3.41</td>
<td>{0,1,2,4,3,5,6}</td>
<td>5.25</td>
</tr>
<tr>
<td>{6,8,9,7}</td>
<td>3.43</td>
<td>{0,1,2,4,3,5,6,8,9,7}</td>
<td>5.57</td>
</tr>
</tbody>
</table>

Figure 9.2. A graphical representation for the fitted values given by the strongly-anti-Robinson matrix in the lower-triangular portion of Table 9.1.

where, as before, PROX is the input proximity matrix \(n \times n\) with a zero main diagonal and a dissimilarity interpretation; INPERM is a given starting permutation of the first \(n\) integers; FIND is the least-squares optimal matrix (with variance-accounted-for of VAF) to PROX and is the sum of the two anti-Robinson matrices TARGONE and TARGTWO based on the two row and column object orderings given by the ending permutations OUTPERMONE and OUTPERMTWO. As before, KBLOCK defines the block size in the use the iterative quadratic assignment routine.

In the example below, the two resulting AR forms are very clearly interpretable as number magnitude and digit structural properties; the VAF is, in effect, 100%.
load number.dat

[find, vaf, targone, targtwo, outpermone, outpermtwo] = ...
biarobfnd(number, 1:10, 1)

find =

Columns 1 through 7

<table>
<thead>
<tr>
<th>0</th>
<th>0.4209</th>
<th>0.5840</th>
<th>0.7090</th>
<th>0.6840</th>
<th>0.8040</th>
<th>0.7865</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4209</td>
<td>0</td>
<td>0.2840</td>
<td>0.3460</td>
<td>0.6460</td>
<td>0.5880</td>
<td>0.7568</td>
</tr>
<tr>
<td>0.5840</td>
<td>0.2840</td>
<td>0</td>
<td>0.3540</td>
<td>0.0588</td>
<td>0.6702</td>
<td>0.4225</td>
</tr>
<tr>
<td>0.7090</td>
<td>0.3460</td>
<td>0.3540</td>
<td>0</td>
<td>0.4130</td>
<td>0.4290</td>
<td>0.3000</td>
</tr>
<tr>
<td>0.6840</td>
<td>0.6460</td>
<td>0.0588</td>
<td>0.4130</td>
<td>0</td>
<td>0.4094</td>
<td>0.3880</td>
</tr>
<tr>
<td>0.8040</td>
<td>0.5880</td>
<td>0.6702</td>
<td>0.4290</td>
<td>0.4094</td>
<td>0</td>
<td>0.3960</td>
</tr>
<tr>
<td>0.7865</td>
<td>0.7568</td>
<td>0.4225</td>
<td>0.3000</td>
<td>0.3880</td>
<td>0.3960</td>
<td>0</td>
</tr>
<tr>
<td>0.9107</td>
<td>0.6300</td>
<td>0.7960</td>
<td>0.5920</td>
<td>0.7420</td>
<td>0.4000</td>
<td>0.4169</td>
</tr>
<tr>
<td>0.8210</td>
<td>0.7975</td>
<td>0.3672</td>
<td>0.7975</td>
<td>0.2460</td>
<td>0.6714</td>
<td>0.3499</td>
</tr>
<tr>
<td>0.8500</td>
<td>0.6250</td>
<td>0.8080</td>
<td>0.2630</td>
<td>0.6829</td>
<td>0.5920</td>
<td>0.2960</td>
</tr>
</tbody>
</table>

Columns 8 through 10

<table>
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<tr>
<th>0.9107</th>
<th>0.8210</th>
<th>0.8500</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6300</td>
<td>0.7975</td>
<td>0.6250</td>
</tr>
<tr>
<td>0.7960</td>
<td>0.3672</td>
<td>0.8080</td>
</tr>
<tr>
<td>0.5920</td>
<td>0.7975</td>
<td>0.2630</td>
</tr>
<tr>
<td>0.7420</td>
<td>0.2460</td>
<td>0.6829</td>
</tr>
<tr>
<td>0.4000</td>
<td>0.6714</td>
<td>0.5920</td>
</tr>
<tr>
<td>0.4169</td>
<td>0.3499</td>
<td>0.2960</td>
</tr>
<tr>
<td>0</td>
<td>0.4000</td>
<td>0.4587</td>
</tr>
<tr>
<td>0.4000</td>
<td>0</td>
<td>0.3922</td>
</tr>
<tr>
<td>0.4587</td>
<td>0.3922</td>
<td>0</td>
</tr>
</tbody>
</table>

vaf =

0.9999

targone =

Columns 1 through 7

<table>
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<tr>
<th>0</th>
<th>0.3406</th>
<th>0.6710</th>
<th>0.6926</th>
<th>0.6956</th>
<th>0.6956</th>
<th>0.8303</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3406</td>
<td>0</td>
<td>0.2018</td>
<td>0.5421</td>
<td>0.5423</td>
<td>0.5880</td>
<td>0.6764</td>
</tr>
<tr>
<td>0.6710</td>
<td>0.2018</td>
<td>0</td>
<td>0.3333</td>
<td>0.3680</td>
<td>0.4662</td>
<td>0.4662</td>
</tr>
</tbody>
</table>
9.6. Representation Through Multiple (Strongly) AR Matrices

\[
\begin{bmatrix}
0.6926 & 0.5421 & 0.3333 & 0 & 0.3093 & 0.3206 & 0.3779 \\
0.6956 & 0.5423 & 0.3680 & 0.3093 & 0 & 0.2055 & 0.3779 \\
0.6956 & 0.5880 & 0.4662 & 0.3206 & 0.2055 & 0 & 0.2876 \\
0.8303 & 0.6764 & 0.4662 & 0.3779 & 0.3779 & 0.2876 & 0 \\
0.8303 & 0.6764 & 0.6764 & 0.6764 & 0.6383 & 0.4675 & 0.3360 \\
0.8303 & 0.7511 & 0.7138 & 0.6764 & 0.6383 & 0.4745 & 0.3366 \\
0.8611 & 0.7943 & 0.7943 & 0.6764 & 0.6690 & 0.4836 & 0.3849 \\
\end{bmatrix}
\]

Columns 8 through 10

\[
\begin{bmatrix}
0.8303 & 0.8303 & 0.8611 \\
0.6764 & 0.7511 & 0.7943 \\
0.6764 & 0.7138 & 0.7943 \\
0.6764 & 0.6764 & 0.6764 \\
0.6383 & 0.6383 & 0.6690 \\
0.4675 & 0.4745 & 0.4836 \\
0.3360 & 0.3366 & 0.3849 \\
0.2243 & 0.3783 & 0 \\
0.3783 & 0.3783 & 0 \\
\end{bmatrix}
\]

targtwo =

Columns 1 through 7

\[
\begin{bmatrix}
0 & -0.3923 & -0.3092 & -0.0093 & 0.0139 & 0.0139 & 0.1211 \\
-0.3923 & 0 & -0.3092 & -0.0116 & 0.0101 & 0.0139 & 0.1037 \\
-0.3092 & -0.3092 & 0 & -0.0870 & -0.0438 & 0.0137 & 0.0207 \\
-0.0093 & -0.0116 & -0.0870 & 0 & -0.0438 & -0.0111 & 0.0164 \\
0.0139 & 0.0101 & -0.0438 & -0.0438 & 0 & -0.0889 & -0.0779 \\
0.0139 & 0.0139 & 0.0137 & -0.0111 & -0.0889 & 0 & -0.4134 \\
0.1211 & 0.1037 & 0.0207 & 0.0164 & -0.0779 & -0.4134 & 0 \\
0.1211 & 0.1037 & 0.0822 & 0.0804 & 0.0804 & -0.1693 & -0.1961 \\
0.1757 & 0.1037 & 0.0822 & 0.0804 & 0.0804 & 0.0804 & -0.0844 \\
0.2039 & 0.2039 & 0.2039 & 0.1084 & 0.1084 & 0.1084 & 0.1084 \\
\end{bmatrix}
\]

Columns 8 through 10

\[
\begin{bmatrix}
0.1211 & 0.1757 & 0.2039 \\
0.1037 & 0.1037 & 0.2039 \\
0.0822 & 0.0822 & 0.2039 \\
0.0804 & 0.0804 & 0.1084 \\
0.0804 & 0.0804 & 0.1084 \\
-0.1693 & 0.0804 & 0.1084 \\
-0.1961 & -0.0844 & 0.1084 \\
\end{bmatrix}
\]
## 9. Anti-Robinson (AR) Matrices for Symmetric Proximity Data

\[
\begin{pmatrix}
0 & -0.1211 & 0 \\
-0.1211 & 0 & -0.0745 \\
0 & -0.0745 & 0
\end{pmatrix}
\]

\[
\text{outpermone} = \\
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}
\]

\[
\text{outpermtwo} = \\
\begin{array}{cccccccccc}
9 & 5 & 3 & 1 & 7 & 10 & 4 & 2 & 8 & 6
\end{array}
\]

For finding multiple SAR forms, \texttt{bisarobfnd.m} has usage syntax

\[
[\text{find, vaf, targone, targtwo, outpermone, outpermtwo}] = \ldots \\
\text{bisarobfnd(prox, inperm, kblock)}
\]

with all the various terms the same as for \texttt{biarobfnd.m} but now for strongly AR (SAR) structures. The example below finds essentially the same representation as above (involving digit magnitude and structure) with a slight drop in the VAF to 99.06%.

\[
[\text{find, vaf, targone, targtwo, outpermone, outpermtwo}] = \ldots \\
\text{bisarobfnd(number, randperm(10), 1)}
\]

\[
\text{find} = \\
\text{Columns 1 through 7}
\]

\[
\begin{array}{cccccccccc}
0 & 0.4210 & 0.5840 & 0.7095 & 0.6838 & 0.8519 & 0.7260 & \\
0.4210 & 0 & 0.2840 & 0.3460 & 0.6461 & 0.5892 & 0.7565 & \\
0.5840 & 0.2840 & 0 & 0.3541 & 0.0590 & 0.6090 & 0.4830 & \\
0.7095 & 0.3460 & 0.3541 & 0 & 0.4131 & 0.4278 & 0.3005 & \\
0.6838 & 0.6461 & 0.0590 & 0.4131 & 0 & 0.4090 & 0.3882 & \\
0.8519 & 0.5892 & 0.6090 & 0.4278 & 0.4090 & 0 & 0.3960 & \\
0.7260 & 0.7565 & 0.4830 & 0.3005 & 0.3882 & 0.3960 & 0 & \\
0.8998 & 0.6153 & 0.8059 & 0.6067 & 0.7286 & 0.4000 & 0.4168 & \\
0.8208 & 0.8246 & 0.3670 & 0.7893 & 0.2460 & 0.6711 & 0.3502 & \\
0.8736 & 0.6250 & 0.7797 & 0.2630 & 0.6965 & 0.5920 & 0.2955 &
\end{array}
\]

\[
\text{Columns 8 through 10}
\]

\[
\begin{array}{cccc}
0.8998 & 0.8208 & 0.8736 & \\
0.6153 & 0.8246 & 0.6250 & \\
0.8059 & 0.3670 & 0.7797 & \\
0.6067 & 0.7893 & 0.2630 &
\end{array}
\]
9.6. Representation Through Multiple (Strongly) AR Matrices

\[ \begin{align*}
0.7286 & \quad 0.2460 & \quad 0.6965 \\
0.4000 & \quad 0.7111 & \quad 0.5920 \\
0.4168 & \quad 0.3502 & \quad 0.2955 \\
0.2286 & \quad 0.2460 & \quad 0.6965 \\
0.4000 & \quad 0.4000 & \quad 0.4590 \\
0.4000 & \quad 0.4000 & \quad 0.3921 \\
0.4590 & \quad 0.3921 & \quad 0
\end{align*} \]

\[ \text{vaf} = 0.9906 \]

\[ \text{targone} = \]

Columns 1 through 7

\[ \begin{align*}
0 & \quad 0.3148 & \quad 0.6038 & \quad 0.6296 & \quad 0.6296 & \quad 0.7457 & \quad 0.7457 \\
0.3148 & \quad 0 & \quad 0.1778 & \quad 0.5201 & \quad 0.5201 & \quad 0.6626 & \quad 0.6626 \\
0.6038 & \quad 0.1778 & \quad 0 & \quad 0.2742 & \quad 0.3230 & \quad 0.5028 & \quad 0.5028 \\
0.6296 & \quad 0.5201 & \quad 0.2742 & \quad 0 & \quad 0.3192 & \quad 0.5012 & \quad 0.5012 \\
0.6296 & \quad 0.5201 & \quad 0.3230 & \quad 0.3192 & \quad 0 & \quad 0.2831 & \quad 0.3340 \\
0.7457 & \quad 0.6626 & \quad 0.5028 & \quad 0.5012 & \quad 0.2831 & \quad 0 & \quad 0.3021 \\
0.7457 & \quad 0.6626 & \quad 0.5028 & \quad 0.5012 & \quad 0.3340 & \quad 0.3021 & \quad 0 \\
0.7936 & \quad 0.7061 & \quad 0.6997 & \quad 0.6974 & \quad 0.6027 & \quad 0.5526 & \quad 0.3229 \\
0.7936 & \quad 0.7061 & \quad 0.6997 & \quad 0.6974 & \quad 0.6027 & \quad 0.5526 & \quad 0.3229 \\
0.7936 & \quad 0.7061 & \quad 0.6997 & \quad 0.6974 & \quad 0.6027 & \quad 0.5527 & \quad 0.4963 \\
\end{align*} \]

Columns 8 through 10

\[ \begin{align*}
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0.7936 & \quad 0.7936 & \quad 0.7936 \\
0.6997 & \quad 0.6997 & \quad 0.6997 \\
0.6974 & \quad 0.6974 & \quad 0.6974 \\
0.6027 & \quad 0.6027 & \quad 0.6027 \\
0.5526 & \quad 0.5526 & \quad 0.5527 \\
0.3229 & \quad 0.3229 & \quad 0.4963 \\
0 & \quad 0.2815 & \quad 0.4197 \\
0.2815 & \quad 0 & \quad 0.3001 \\
0.4197 & \quad 0.3001 & \quad 0 \\
\end{align*} \]

\[ \text{targtwo} = \]

Columns 1 through 7
Chapter 9. Anti-Robinson (AR) Matrices for Symmetric Proximity Data

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1 2 3 4 5 6 7 8 9 10
```

```
outpermtwo =

5 9 3 1 7 10 4 2 8 6
```
Chapter 10

Circular-Anti-Robinson (CAR) Matrices for Symmetric Proximity Data

In the approximation of a proximity matrix $P$ by one that is row/column reorderable to an AR form, the interpretation of the fitted matrix in general had to be carried out by identifying a set of subsets through an increasing threshold variable; each of the subsets contained objects that were contiguous with respect to a given linear ordering along a continuum, and had a diameter defined by the maximum fitted value within the subset. To provide a further representation depicting the fitted values as lengths of paths in a graph, an approximation was sought that satisfied
the additional constraints of an SAR matrix; still, the subsets thus identified had to contain objects contiguous with respect to a linear ordering. As one possible generalization of both the AR and SAR constraints, we can define what will be called circular anti-Robinson (CAR) and circular strongly-anti-Robinson (CSAR) forms that allow the subsets identified from increasing a threshold variable to be contiguous with respect to a circular ordering of the objects around a closed continuum. Approximation matrices that are row/column reorderable to display an AR or SAR form respectively will also be (trivially) row/column reorderable to display what is formally characterized below as a CAR or a CSAR form, but not conversely. (Historically, there is a large literature on the possibility of circular structures emerging from and being identifiable in a given proximity matrix. For a variety of references, the reader is referred to the American Psychological Association sponsored volume edited by Plutchik and Conte (1997), or the discussion of metric circular unidimensional scaling from Part I, Chapter 3, and in Hubert, Arabie, and Meulman [1997]. The extension of CAR forms to those that are also CSAR, however, has apparently not been a topic discussed in the literature before the appearance of Hubert, Arabie, and Meulman [1998]; this latter source forms the basis for much of the present chapter.)

To be explicit, an arbitrary symmetric matrix \( Q = \{q_{ij}\} \), where \( q_{ii} = 0 \) for \( 1 \leq i, j \leq n \), is said to be row/column reorderable to a circular anti-Robinson form (or, for short, \( Q \) is a circular anti-Robinson (CAR) matrix) if there exists a permutation, \( \rho(\cdot) \), on the first \( n \) integers such that the reordered matrix \( Q_{\rho} = \{q_{\rho(i)\rho(j)}\} \) satisfies the conditions given in (II):

\[
\text{(II): for } 1 \leq i \leq n-3, \text{ and } i + 1 < j \leq n - 1, \text{ then}
\]

\[
\begin{align*}
&\text{if } q_{\rho(i+1)\rho(j)} \leq q_{\rho(i)\rho(j+1)}, \text{ then} \\
&q_{\rho(i+1)\rho(j)} \leq q_{\rho(i)\rho(j)} \text{ and } q_{\rho(i+1)\rho(j)} \leq q_{\rho(i+1)\rho(j+1)}; \\
&\text{if } q_{\rho(i+1)\rho(j)} \geq q_{\rho(i)\rho(j+1)}, \text{ then} \\
&q_{\rho(i)\rho(j)} \geq q_{\rho(i)\rho(j+1)} \text{ and } q_{\rho(i+1)\rho(j+1)} \geq q_{\rho(i)\rho(j+1)}, \\
&\text{and, for } 2 \leq i \leq n - 2, \\
&\text{if } q_{\rho(i+1)\rho(n)} \leq q_{\rho(i)\rho(1)}, \text{ then} \\
&q_{\rho(i)\rho(n)} \leq q_{\rho(i)\rho(n)} \text{ and } q_{\rho(i+1)\rho(n)} \leq q_{\rho(i+1)\rho(1)}; \\
&\text{if } q_{\rho(i+1)\rho(n)} \geq q_{\rho(i)\rho(1)}, \text{ then} \\
&q_{\rho(i)\rho(n)} \geq q_{\rho(i)\rho(1)} \text{ and } q_{\rho(i+1)\rho(1)} \geq q_{\rho(i)\rho(1)}.
\end{align*}
\]

Interpretatively, within each row of \( Q_{\rho} \) moving to the right from the main diagonal and then wrapping back around to re-enter the same row from the left, the entries never decrease until a maximum is reached and then never increase moving away from the maximum until the main diagonal is again reached. Given the symmetry of \( P \), a similar pattern of entries would be present within each column as well. As noted above, any AR matrix is CAR but not conversely.

In analogy to the SAR conditions that permit graphical representation, a symmetric matrix \( Q \) is said to be row/column reorderable to a circular strongly-anti-Robinson form (or, for short, \( Q \) is a circular strongly-anti-Robinson (CSAR) matrix) if there exists a permutation, \( \rho(\cdot) \), on the first \( n \) integers such that the reordered matrix \( Q_{\rho} = \{q_{\rho(i)\rho(j)}\} \) satisfies the conditions given by (II), and
for \(1 \leq i \leq n - 3\), and \(i + 1 < j \leq n - 1\),

if \(q_{\rho(i)\rho(j)} + 1 \leq q_{\rho(i)\rho(j)} + 1\), then \(q_{\rho(i)\rho(j)} = q_{\rho(i)\rho(j)}\) implies \(q_{\rho(i)\rho(j)} = q_{\rho(i)\rho(j)}\); and \(q_{\rho(i)\rho(j)} = q_{\rho(i)\rho(j)}\) implies \(q_{\rho(i)\rho(j)} = q_{\rho(i)\rho(j)}\);

if \(q_{\rho(i)\rho(j)} \geq q_{\rho(i)\rho(j)} + 1\), then \(q_{\rho(i)\rho(j)} = q_{\rho(i)\rho(j)} + 1\) implies \(q_{\rho(i)\rho(j)} = q_{\rho(i)\rho(j)}\); and \(q_{\rho(i)\rho(j)} + 1 = q_{\rho(i)\rho(j)}\) implies \(q_{\rho(i)\rho(j)} + 1 = q_{\rho(i)\rho(j)}\);

and for \(2 \leq i \leq n - 2\),

if \(q_{\rho(i)\rho(n)} \leq q_{\rho(i)\rho(n)}\), then \(q_{\rho(i)\rho(n)} = q_{\rho(i)\rho(n)}\) implies \(q_{\rho(i)\rho(n)} = q_{\rho(i)\rho(n)}\), and \(q_{\rho(i)\rho(n)} = q_{\rho(i)\rho(n)}\) implies \(q_{\rho(i)\rho(n)} = q_{\rho(i)\rho(n)}\);

if \(q_{\rho(i)\rho(n)} \geq q_{\rho(i)\rho(n)}\), then \(q_{\rho(i)\rho(n)} = q_{\rho(i)\rho(n)}\) implies \(q_{\rho(i)\rho(n)} = q_{\rho(i)\rho(n)}\), and \(q_{\rho(i)\rho(n)} = q_{\rho(i)\rho(n)}\) implies \(q_{\rho(i)\rho(n)} = q_{\rho(i)\rho(n)}\).

Again, the imposition of the stronger CSAR conditions avoids the type of graphical anomaly present in Figure 9.1(b) but now in the context of a CAR matrix — when two fitted values that are adjacent within a row are equal, the fitted values in the same two adjacent columns must also be equal for a row that is either its immediate predecessor (if \(q_{\rho(i)\rho(j)} \leq q_{\rho(j)\rho(n)}\)), or successor (if \(q_{\rho(i)\rho(j)} \geq q_{\rho(j)\rho(n)}\)); a similar condition is imposed when two fitted values that are adjacent within a column are equal. As noted, any SAR matrix is CSAR but not conversely.

The computational strategy we suggest for identifying a best-fitting CAR or CSAR approximation matrix is based on an initial circular unidimensional scaling obtained through the optimization strategy developed by Hubert, Arabie, and Meulman (1997) that is reviewed in Part I, Chapter 3. Specifically, we first institute a combination of combinatorial search for good matrix reorderings and heuristic iterative projection to locate the points of inflection when minimum distance calculations change directionality around a closed circular structure. Approximation matrices to \(P\) are found through a least-squares loss criterion, and they have the parameterized form

\[ Q_\rho = \{ \min(|x_{\rho(j)} - x_{\rho(i)}|, x_0 - |x_{\rho(j)} - x_{\rho(i)}|) + c \}, \]

where \(c\) is an estimated additive constant, \(x_{\rho(1)} \leq x_{\rho(2)} \leq \cdots \leq x_{\rho(n)} \leq x_0\), and the last coordinate, \(x_0\), is the circumference of the circular structure. Based on the inequality constraints implied by such a selection of coordinates, a CAR approximation matrix can be fitted to \(P\) directly; then, beginning with this latter CAR approximation, the identification and imposition of CSAR constraints proceeds through the heuristic use of iterative projection, directly analogous to the way SAR constraints in the linear ordering context were identified and fitted, beginning with a best approximation matrix satisfying just the AR restrictions.

### 10.1 Fitting a Given CAR Matrix in the \(L_2\)-Norm

The function m-file, `cirarobfit.m`, fits a circular anti-Robinson (CAR) matrix using iterative projection to a symmetric proximity matrix in the \(L_2\)-norm. Usage syntax is

```matlab
[fit, vaf] = cirarobfit(prox, inperm, targ)
```
where \textbf{PROX} is the input proximity matrix \((n \times n\) with a zero main diagonal and a dissimilarity interpretation); \textbf{INPERM} is a given permutation of the first \(n\) integers (around a circle); \textbf{TARG} is a given \(n \times n\) matrix having the circular anti-Robinson form that guides the direction in which distances are taken around the circle. The matrix \textbf{FIT} is the least-squares optimal approximation (with variance-accounted-for of \textbf{VAF}) to \textbf{PROX} having an circular anti-Robinson form for the row and column object ordering given by \textbf{INPERM}.

A recording of a MATLAB session follows that uses the \texttt{number.dat} data file; an equally-spaced circular anti-Robinson matrix \texttt{targcircular} obtained from the utility m-file \texttt{targcir.m} first introduced in Part I; and the identity permutation for the objects around the circular structure. The fitted CAR matrix thus identified in this way has a \textbf{VAF} of 64.37%.

```matlab
load number.dat
targcircular = targcir(10);
[fit vaf] = cirarobfit(number,1:10,targcircular)
```

```latex
\begin{verbatim}
fit =

Columns 1 through 7

0   0.4210  0.5840  0.6510  0.6835  0.8040  0.7730
0.4210   0  0.2840  0.3460  0.6170  0.6170  0.7730
0.5840  0.2840   0  0.2753  0.2753  0.5460  0.5460
0.6510  0.3460  0.2753   0  0.2753  0.3844  0.3844
0.6835  0.6170  0.2753  0.2753   0  0.3844  0.3844
0.8040  0.6170  0.5460  0.3844  0.3844   0  0.3844
0.7730  0.7730  0.5460  0.3844  0.3844  0.3844   0
0.7695  0.7695  0.7960  0.5920  0.5530  0.4000  0.3857
0.6597  0.6597  0.6597  0.8040  0.5530  0.5530  0.3857
0.6510  0.6510  0.6510  0.6510  0.6835  0.5920  0.3857

Columns 8 through 10

0.7695  0.6597  0.6510
0.7695  0.6597  0.6510
0.7960  0.6597  0.6510
0.5920  0.8040  0.6510
0.5530  0.5530  0.6835
0.4000  0.5530  0.5920
0.3857  0.3857  0.3857
0  0.3857  0.3857
0.3857  0  0.3857
0.3857  0.3857  0
\end{verbatim}
```
10.2 Finding a CAR Matrix in the $L_2$-Norm

The m-file, *cirarobfnd.m*, is our suggested strategy for identifying a best-fitting CAR matrix for a symmetric proximity matrix in the $L_2$-norm based on a permutation that is initially identified through the use of iterative quadratic assignment. Based on an equally-spaced circular target matrix, *order.m* is first invoked to obtain a good (circular) permutation, which in turn is then used to construct a new circular target matrix with *cirfit.m*. (We will mention here but not illustrate with an example, an alternative to the use of *cirarobfnd.m* called *cirarobfnd_ac.m*; the latter m-file has the same syntax as *cirarobfnd.m* but uses *cirfitac.m* rather than *cirfit.m* internally to obtain the new circular target matrices.) The final output is generated from *cirarobfit.m* that no better permutation can be identified using the newer circular target matrix. The usage syntax for *cirarobfnd.m* is as follows:

```
[fit, vaf, outperm] = cirarobfnd(prox, inperm, kblock)
```

where *PROX* is the input proximity matrix ($n \times n$ with a zero main diagonal and a dissimilarity interpretation); *INPERM* is a given starting permutation (assumed to be around the circle) of the first $n$ integers; *FIT* is the least-squares optimal matrix (with variance-accounted-for of VAF) to *PROX* having a circular anti-Robinson form for the row and column object ordering given by the concluding permutation *OUTPERM*. Again, *KBLOCK* defines the block size in the use of the iterative quadratic assignment routine.

An example of the use of *cirarobfnd.m* is given below that seems to lead to a circular ordering best interpreted according to the structural properties of the digits. This solution is only one of several local optima identifiable by repeated application of the routine using other random starting permutations. In general, the different local optima observed differ in the way the odd digits, $\{3, 5, 7, 9\}$, and the even digits, $\{2, 4, 6, 8\}$, are ordered within these sets when moving clockwise around a circular structure. Explicitly, all local optima had a general structure of $\rightarrow 0 \rightarrow 1 \rightarrow \{3, 5, 7, 9\} \rightarrow \{2, 4, 6, 8\} \rightarrow$, but with some variation in order within the odd and even digits. For example, the CAR matrix given below uses the odd digits as $\rightarrow 3 \rightarrow 5 \rightarrow 9 \rightarrow 7 \rightarrow$ and the even digits as $\rightarrow 6 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow$.

```
[fit, vaf, outperm] = cirarobfnd(number, randperm(10), 3)
```

```
fit =

Columns 1 through 7

0 0.3460 0.5315 0.5315 0.6069 0.8040 0.4460

0.3460 0 0.4210 0.4340 0.6069 0.7895 0.7895
```
### 10.3 Finding a Circular Strongly-Anti-Robinson (CSAR) Matrix in the $L_2$-Norm

The two m-functions, `cirsarobfit.m` and `cirsarobfnd.m`, are direct analogues of `cirarobfit.m` and `cirarobfnd.m`, respectively, but are concerned with fitting and finding strongly circular-anti-Robinson forms (also, we mention but do not illustrate, the m-file `cirsarobfnd_ac.m` which uses `cirarobfnd_ac.m` to obtain the initial CAR matrix that is then strengthened into one that is CSAR). The syntax for `cirsarobfit.m`, which fits a circular strongly-anti-Robinson matrix using iterative projection to a symmetric proximity matrix in the $L_2$-norm, is

$$ [\text{fit}, \text{vaf}] = \text{cirsarobfit} (\text{prox}, \text{inperm}, \text{targ}) $$

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Columns 8 through 10

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vaf =

0.8128

outperm =

4 2 1 3 5 9 7 8 10 6
10.3. Finding a Circular Strongly-Anti-Robinson (CSAR) Matrix in the $L_2$-Norm

where, again, PROX is the input proximity matrix ($n \times n$ with a zero main diagonal and a dissimilarity interpretation); INPERM is a given permutation of the first $n$ integers; TARG is a given $n \times n$ matrix having the circular anti-Robinson form that guides the direction in which distances are taken around the circle. FIT is the least-squares optimal matrix (with variance-accounted-for of VAF) to PROX having a strongly circular-anti-Robinson form for the row and column object ordering given by INPERM.

An example follows using the same identity permutation as in fitting a CAR form with cirarobfit.m; as might be expected from using the more restrictive CSAR form, the variance-accounted-for drops to .4501 from .6437.

$[fit, vaf] = 
\text{cirsarobfit(number,1:10,targcircular)}$

fit =

Columns 1 through 7

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vaf =

0.4501
The m-function, `cirsarobfnd.m`, finds and fits a CSAR matrix using iterative projection to a symmetric proximity matrix in the $L_2$-norm based on a permutation identified through the use of iterative quadratic assignment. It has the expected syntax

$$[\text{fit}, \text{vaf}, \text{outperm}] = \text{cirsarobfnd}(\text{prox}, \text{inperm, kblock})$$

where, again, $\text{PROX}$ is the input proximity matrix ($n \times n$ with a zero main diagonal and a dissimilarity interpretation); $\text{INPERM}$ is a given starting permutation of the first $n$ integers; $\text{FIT}$ is the least-squares optimal matrix (with variance-accounted-for of VAF) to $\text{PROX}$ having a circular strongly-anti-Robinson form for the row and column object ordering given by the ending permutation $\text{OUTPERM}$. As usual, $\text{KBLOCK}$ defines the block size in the use the iterative quadratic assignment routine. (Analogous to the last section, and as noted above, an alternative to $\text{cirsarobfnd.m}$ is available, called $\text{cirsarobfnd_ac.m}$ that uses $\text{cirfitac.m}$ to obtain the circular target matrices.)

In the MATLAB recording below, and starting from a random permutation, a circular strongly anti-Robinson form was found with a VAF of .7296 (again, this represents an expected drop from the value of .8119 for the CAR form — this result is also listed below)).

$$[\text{fit}, \text{vaf}, \text{outperm}] = \text{cirsarobfnd}(\text{number,randperm(10)}, 2)$$

target =

Columns 1 through 6

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Columns 7 through 10

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<td></td>
<td></td>
</tr>
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</table>
10.3. Finding a Circular Strongly-Anti-Robinson (CSAR) Matrix in the \( L_2 \)-Norm

\[
\begin{pmatrix}
0.4340 & 0 & 0.4210 & 0.5315 \\
0.4340 & 0.4210 & 0 & 0.3460 \\
0.5315 & 0.5315 & 0.3460 & 0 \\
\end{pmatrix}
\]

\[vaf = 0.8119\]

\[\text{outperm} = \]

\[6 \quad 10 \quad 8 \quad 7 \quad 9 \quad 5 \quad 3 \quad 1 \quad 2 \quad 4\]

\[\text{fit} = \]

Columns 1 through 6

\[
\begin{pmatrix}
0 & 0.4246 & 0.4246 & 0.4246 & 0.7304 & 0.7304 \\
0.4246 & 0 & 0.3907 & 0.3907 & 0.3960 & 0.7304 \\
0.4246 & 0.3907 & 0 & 0.3907 & 0.3960 & 0.7304 \\
0.4246 & 0.3907 & 0.3907 & 0 & 0.3500 & 0.3880 \\
0.7304 & 0.3960 & 0.3960 & 0.3500 & 0 & 0.2460 \\
0.7304 & 0.7304 & 0.7304 & 0.3880 & 0.2460 & 0 \\
0.7304 & 0.7304 & 0.7304 & 0.4210 & 0.3670 & 0.0590 \\
0.7304 & 0.7304 & 0.7304 & 0.7304 & 0.7304 & 0.5810 \\
0.7304 & 0.7304 & 0.7304 & 0.7304 & 0.7304 & 0.5810 \\
0.4246 & 0.4246 & 0.4246 & 0.4246 & 0.7304 & 0.5810 \\
\end{pmatrix}
\]

Columns 7 through 10

\[
\begin{pmatrix}
0.7304 & 0.7304 & 0.7304 & 0.4246 \\
0.7304 & 0.7304 & 0.7304 & 0.4246 \\
0.7304 & 0.7304 & 0.7304 & 0.4246 \\
0.4210 & 0.7304 & 0.7304 & 0.4246 \\
0.3670 & 0.7304 & 0.7304 & 0.7304 \\
0.0590 & 0.5810 & 0.5810 & 0.5810 \\
0 & 0.4340 & 0.4340 & 0.5315 \\
0.4340 & 0 & 0.4210 & 0.5315 \\
0.4340 & 0.4210 & 0 & 0.3460 \\
0.5315 & 0.5315 & 0.3460 & 0 \\
\end{pmatrix}
\]

\[vaf = \]
0.7296

\text{outperm} =

\begin{align*}
6 & \quad 10 & \quad 8 & \quad 7 & \quad 9 & \quad 5 & \quad 3 & \quad 1 & \quad 2 & \quad 4
\end{align*}

10.4 Graphically Representing CSAR Structures

As in the case of an AR or SAR matrix, the interpretation of the structure that may be represented by a CAR or CSAR matrix could proceed by first identifying those subsets and their diameters that emerge by increasing a threshold variable from the smallest fitted value. And in the case of a more restrictive CSAR matrix, this collection of subsets and their diameters can then be displayed by a graph where minimum length paths reconstruct the fitted values. To illustrate this graphical possibility on the transformed \text{number.dat} to mean 4.0 and variance 1.0 given in Hubert, Arabie, and Meulman (1998) — and used earlier to show the graphical representation of an SAR matrix — the fifteen (nonredundant) subsets identified from the CSAR matrix present in Table 10.2 are listed in Table 10.1 according to increasing diameter. Here, the structural properties of the digits are apparent (e.g., various subsets of the odd or even digits, or those that are multiples or powers of 2 or of 3), but some magnitude adjacencies can also be noted (e.g., \{6, 7, 8, 9\}, or subsets of \{0, 1, 2\}). The graph adhering to the CSAR restrictions is given in Figure 10.1 and again minimum path lengths (that proceed up from a terminal node to an internal node and then back down to the other terminal node) can be used to reconstruct the fitted values in \(Q\).

In addition to searching for a best-fitting CSAR matrix directly, we might comment that the type of indirect approach mentioned in the introduction for the case of SAR approximations could also be considered, although we will not go into the details here. For example, based on a best-fitting CAR matrix, the additional constraints of a circular unidimensional scale could be identified and then imposed (in fact, this is our starting place in first obtaining the CAR approximation); or those of an ultrametric (which would lead to an SAR matrix that is trivially CSAR as well); or possibly, a collection of additive tree restrictions could be identified. In all cases, CSAR approximations would be automatically obtained.

10.5 Representation Through Multiple (Strongly) CAR Matrices

Just as we discussed in Section 9.6 the representing of proximity matrices through multiple (strongly) AR matrices, representations of a proximity matrix by a single (strongly) circular-anti-Robinson structure extends easily to the additive use of multiple matrices. The m-function, \text{bicirarobfnd.m}, fits the sum of two circular-anti-Robinson matrices using iterative projection to a symmetric proximity matrix in the \(L_2\)-norm based on permutations identified through the use of iterative quadratic
10.5. Representation Through Multiple (Strongly) CAR Matrices

Table 10.1. The fifteen (nonredundant) subsets listed according to increasing diameter values are contiguous in the circular object ordering used to display the CSAR entries in Table 10.2.

<table>
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<th>diameter</th>
<th>subset</th>
<th>diameter</th>
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<td>{3, 5, 9, 7, 6}</td>
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<td>{2, 0, 1}</td>
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<td>3.95</td>
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<td>{4, 2, 0, 1, 3}</td>
<td>4.20</td>
</tr>
<tr>
<td>{9, 7, 6}</td>
<td>3.26</td>
<td>{0, 1, 3, 5, 9, 7, 6, 8, 4, 2}</td>
<td>4.93</td>
</tr>
<tr>
<td>{9, 7, 6, 8}</td>
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Table 10.2. A circular strongly-anti-Robinson order-constrained least-squares approximations to the digit proximity data of Shepard et al. (1975).

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<th>7</th>
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<td>1</td>
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<td>x</td>
<td>3.04</td>
<td>4.93</td>
<td>4.93</td>
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<td>x</td>
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</table>

assignment. The syntax usage is

```
[find, vaf, targone, targtwo, outpermone, outpermtwo] = ...
bicirarobfnd(prox, inperm, kblock)
```

where, as before, PROX is the input proximity matrix \((n \times n)\) with a zero main diagonal and a dissimilarity interpretation; INPERM is a given initial permutation of the first \(n\) integers; FIND is the least-squares optimal matrix (with variance-accounted-for of VAF) to PROX and is the sum of the two circular-anti-Robinson matrices TARGONE and TARGTWO based on the two row and column object orderings given by the final permutations OUTPERMONE and OUTPERMTWO. As before, KBLOCK defines the block size in the use of the iterative quadratic assignment routine.
Figure 10.1. A graphical representation for the fitted values given by the circular strongly-anti-Robinson matrix in the lower-triangular portion of Table 10.2 (VAF = 72.96%). Note that digit 3 is placed both in the first and the last positions in the ordering of the objects with the implication that the sequence continues in a circular manner. This circularity is indicated by the curved dashed line.

>> [find, vaf, targone, targtwo, outpermone, outpermtwo] = ...  
bicirobfnd(number, randperm(10), 1)

find = 

Columns 1 through 6

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<tr>
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<td>0.6460</td>
<td>0.0380</td>
<td>0.4340</td>
<td>0</td>
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<td></td>
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<tr>
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<td>0.3317</td>
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</tr>
<tr>
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<td>0.6380</td>
<td>0.8131</td>
<td>0.5750</td>
<td>0.7418</td>
<td>0.4000</td>
<td></td>
</tr>
<tr>
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<td>0.7926</td>
<td>0.3881</td>
<td>0.7841</td>
<td>0.2460</td>
<td>0.6710</td>
<td></td>
</tr>
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<td>0.2631</td>
<td>0.6830</td>
<td>0.5899</td>
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</tbody>
</table>

Columns 7 through 10

<table>
<thead>
<tr>
<th></th>
<th>0.7871</th>
<th>0.9090</th>
<th>0.8210</th>
<th>0.8521</th>
</tr>
</thead>
</table>

>>
10.5. Representation Through Multiple (Strongly) CAR Matrices

\[
\begin{array}{cccccc}
0.7580 & 0.6380 & 0.7926 & 0.6380 \\
0.4208 & 0.8131 & 0.3881 & 0.7841 \\
0.3317 & 0.5750 & 0.7841 & 0.2631 \\
0.3665 & 0.7418 & 0.2460 & 0.6830 \\
0.3963 & 0.4000 & 0.6710 & 0.5899 \\
0 & 0.4176 & 0.3500 & 0.2960 \\
0.4176 & 0 & 0.4000 & 0.4590 \\
0.3500 & 0.4000 & 0 & 0.3920 \\
0.2960 & 0.4590 & 0.3920 & 0
\end{array}
\]

vaf =

0.9955

targone =

Columns 1 through 6

\[
\begin{array}{cccccc}
0 & 0.0858 & 0.0858 & 0.3086 & 0.4576 & 0.4576 \\
0.0858 & 0 & 0.0096 & 0.2443 & 0.2443 & 0.3863 \\
0.0858 & 0.0096 & 0 & 0.2133 & 0.2391 & 0.2391 \\
0.3086 & 0.2443 & 0.2133 & 0 & 0.0994 & 0.1207 \\
0.4576 & 0.2443 & 0.2391 & 0.0994 & 0 & 0.1207 \\
0.4576 & 0.3863 & 0.2391 & 0.1207 & 0.1207 & 0 \\
0.4818 & 0.4818 & 0.3631 & 0.2195 & 0.2195 & 0.2195 \\
0.4818 & 0.4818 & 0.4818 & 0.2195 & 0.2195 & 0.2195 \\
0.3153 & 0.4902 & 0.4902 & 0.4902 & 0.4711 & 0.3356 \\
0.3153 & 0.4361 & 0.4628 & 0.4902 & 0.7370 & 0.7185
\end{array}
\]

Columns 7 through 10

\[
\begin{array}{cccccc}
0.4818 & 0.4818 & 0.3153 & 0.3153 \\
0.4818 & 0.4818 & 0.4902 & 0.4361 \\
0.3631 & 0.4818 & 0.4902 & 0.4628 \\
0.2195 & 0.2195 & 0.4902 & 0.4902 \\
0.2195 & 0.2195 & 0.4711 & 0.7370 \\
0.2195 & 0.2195 & 0.3356 & 0.7185 \\
0 & -0.0393 & 0.3356 & 0.4818 \\
-0.0393 & 0 & 0.3356 & 0.4818 \\
0.3356 & 0.3356 & 0 & 0.2371 \\
0.4818 & 0.4818 & 0.2371 & 0
\end{array}
\]

targtwo =
Columns 1 through 6

\[
\begin{array}{cccccc}
0 & 0.0765 & 0.1367 & 0.2969 & 0.2969 & 0.2969 \\
0.0765 & 0 & 0.0289 & 0.2395 & 0.3704 & 0.3704 \\
0.1367 & 0.0289 & 0 & 0.1609 & 0.3582 & 0.4319 \\
0.2969 & 0.2395 & 0.1609 & 0 & 0.1905 & 0.2793 \\
0.2969 & 0.3704 & 0.3582 & 0.1905 & 0 & 0.0670 \\
0.2969 & 0.3704 & 0.4319 & 0.2793 & 0.0670 & 0 \\
0.2678 & 0.3024 & 0.3024 & 0.3024 & 0.1839 & 0.1169 \\
0.1122 & 0.3024 & 0.3024 & 0.3555 & 0.2480 & 0.1959 \\
0.1122 & 0.3024 & 0.3024 & 0.3555 & 0.2480 & 0.1959 \\
0.1122 & 0.2012 & 0.2364 & 0.3555 & 0.2480 & 0.1959 \\
\end{array}
\]

Columns 7 through 10

\[
\begin{array}{cccc}
0.2678 & 0.1122 & 0.1122 & 0.1122 \\
0.3024 & 0.3024 & 0.3024 & 0.2012 \\
0.3024 & 0.3024 & 0.3024 & 0.2364 \\
0.3024 & 0.3555 & 0.3555 & 0.3555 \\
0.1839 & 0.2480 & 0.2480 & 0.2480 \\
0.1169 & 0.1959 & 0.1959 & 0.1959 \\
-0.0105 & -0.0105 & -0.0105 & 0.1558 \\
-0.0105 & 0 & -0.1278 & -0.0478 \\
-0.0105 & -0.1278 & 0 & -0.0478 \\
0.1558 & -0.0478 & -0.0478 & 0 \\
\end{array}
\]

\[
\text{outpermone = }
\begin{array}{cccccccc}
3 & 5 & 9 & 7 & 6 & 8 & 10 & 4 & 2 & 1
\end{array}
\]

\[
\text{outpermtwo = }
\begin{array}{cccccccc}
7 & 10 & 9 & 8 & 1 & 6 & 2 & 3 & 4 & 5
\end{array}
\]

For finding multiple CSAR forms, bicirsarobfnd.m has usage syntax

\[
\text{[find,vaf,targone,targtwo,outpermone,outpermtwo] = ...}
\]

bicirsarobfnd(prox,inperm,kblock)

with all the various terms the same as for bicirarobfnd.m but now for strongly CAR (CSAR) structures. The example below finds essentially the same representation as above (involving digit magnitude and structure) with a slight drop in the VAF from 99.55% for CAR to 91.06% for CSAR.

\[
\gg \text{[find,vaf,targone,targtwo,outpermone,outpermtwo] = ...}
\]
bicirsarobfnd(number,randperm(10),1)

find =

Columns 1 through 6

<p>| | | | | |</p>
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<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0.4212</td>
<td>0.6464</td>
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<tr>
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<td>0.3284</td>
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<td>0</td>
<td>0.1111</td>
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<tr>
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<td>0.5111</td>
<td>0</td>
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<td>0.8040</td>
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<td>0.6682</td>
<td>0.3505</td>
<td>0</td>
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<tr>
<td>0.8420</td>
<td>0.7215</td>
<td>0.4802</td>
<td>0.4027</td>
<td>0.4493</td>
</tr>
<tr>
<td>0.8420</td>
<td>0.7215</td>
<td>0.7215</td>
<td>0.6565</td>
<td>0.6906</td>
</tr>
<tr>
<td>0.8420</td>
<td>0.7215</td>
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<td>0.7204</td>
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Columns 7 through 10

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<td>0.7215</td>
<td>0.7215</td>
<td></td>
</tr>
<tr>
<td>0.4802</td>
<td>0.7215</td>
<td>0.3041</td>
<td>0.7204</td>
<td></td>
</tr>
<tr>
<td>0.4027</td>
<td>0.6565</td>
<td>0.7204</td>
<td>0.2630</td>
<td></td>
</tr>
<tr>
<td>0.4493</td>
<td>0.6906</td>
<td>0.2732</td>
<td>0.6895</td>
<td></td>
</tr>
<tr>
<td>0.3718</td>
<td>0.4000</td>
<td>0.6895</td>
<td>0.5540</td>
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<td>0</td>
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<td>0.2292</td>
<td>0.3339</td>
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<td>0</td>
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<td>0.4055</td>
<td></td>
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<td>0.2292</td>
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vaf =

0.9106

targone =

Columns 1 through 6

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<tbody>
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<td></td>
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<td>0.3924</td>
<td>0.6326</td>
<td>0.6326</td>
<td>0.6326</td>
</tr>
<tr>
<td>0.3924</td>
<td>0</td>
<td>0.3149</td>
<td>0.3149</td>
<td>0.4970</td>
<td>0.5686</td>
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<tr>
<td>0.6326</td>
<td>0.3149</td>
<td>0</td>
<td>0.3149</td>
<td>0.4970</td>
<td>0.5686</td>
</tr>
<tr>
<td>0.6326</td>
<td>0.3149</td>
<td>0.3149</td>
<td>0</td>
<td>0.1752</td>
<td>0.5686</td>
</tr>
<tr>
<td>0.6326</td>
<td>0.4970</td>
<td>0.4970</td>
<td>0.1752</td>
<td>0</td>
<td>0.5686</td>
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<tr>
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</tr>
</tbody>
</table>
Chapter 10. Circular-Anti-Robinson (CAR) Matrices

\[
\begin{bmatrix}
0.6337 & 0.6337 & 0.6337 & 0.6337 & 0.6337 & 0.6337 \\
0.6337 & 0.6337 & 0.6337 & 0.6337 & 0.6337 & 0.6337 \\
0.2162 & 0.3924 & 0.6326 & 0.6326 & 0.6326 & 0.6337 \\
0.2162 & 0.3924 & 0.6326 & 0.6326 & 0.6326 & 0.6337 \\
\end{bmatrix}
\]

Columns 7 through 10

\[
\begin{bmatrix}
0.6337 & 0.6337 & 0.2162 & 0.2162 \\
0.6337 & 0.6337 & 0.3924 & 0.3924 \\
0.6337 & 0.6337 & 0.6326 & 0.6326 \\
0.6337 & 0.6337 & 0.6326 & 0.6326 \\
0.6337 & 0.6337 & 0.6326 & 0.6326 \\
0.6337 & 0.6337 & 0.6337 & 0.6337 \\
0 & 0.4085 & 0.6337 & 0.6337 \\
0.4085 & 0 & 0.6337 & 0.6337 \\
0.6337 & 0.6337 & 0 & 0.2162 \\
0.6337 & 0.6337 & 0.2162 & 0 \\
\end{bmatrix}
\]

targtwo =

Columns 1 through 6

\[
\begin{bmatrix}
0 & -0.2236 & 0.0570 & 0.0570 & 0.0570 & 0.0570 \\
-0.2236 & 0 & -0.1686 & 0.0570 & 0.0570 & 0.0570 \\
0.0570 & -0.1686 & 0 & -0.1632 & -0.1632 & -0.1632 \\
0.0570 & 0.0570 & -0.1632 & 0 & -0.1632 & -0.1632 \\
0.0570 & 0.0570 & -0.1632 & -0.1632 & 0 & -0.1632 \\
0.0570 & 0.0570 & -0.1632 & -0.1632 & -0.1632 & 0 \\
0.0503 & 0.1703 & 0.2083 & 0.2083 & 0.2083 & 0.2083 \\
-0.1215 & 0.0356 & 0.0878 & 0.0878 & 0.0878 & 0.0878 \\
-0.1215 & 0.0356 & 0.0878 & 0.0878 & 0.0878 & 0.0878 \\
-0.1215 & 0.0356 & 0.0878 & 0.0878 & 0.0878 & 0.0878 \\
\end{bmatrix}
\]

Columns 7 through 10

\[
\begin{bmatrix}
0.0503 & -0.1215 & -0.1215 & -0.1215 \\
0.1703 & 0.0356 & 0.0356 & 0.0356 \\
0.2083 & 0.0878 & 0.0878 & 0.0878 \\
0.2083 & 0.0878 & 0.0878 & 0.0878 \\
0.2083 & 0.0878 & 0.0878 & 0.0878 \\
0.2083 & 0.0878 & 0.0878 & 0.0878 \\
0 & 0.0127 & 0.0127 & 0.0127 \\
0.0127 & 0 & -0.3053 & -0.3053 \\
0.0127 & -0.3053 & 0 & -0.3053 \\
0.0127 & -0.3053 & -0.3053 & 0 \\
\end{bmatrix}
\]
10.5. Representation Through Multiple (Strongly) CAR Matrices

\[
\begin{align*}
\text{outpermone} &= \begin{array}{cccccccc}
5 & 7 & 6 & 4 & 10 & 8 & 2 & 1 & 9 & 3 \\
\end{array} \\
\text{outpermtwo} &= \begin{array}{cccccccc}
5 & 6 & 8 & 9 & 10 & 7 & 1 & 4 & 3 & 2 \\
\end{array}
\end{align*}
\]
Chapter 10. Circular-Anti-Robinson (CAR) Matrices
Chapter 11

Anti-Robinson (AR) Matrices for Two-Mode Proximity Data

In direct analogy to the extensions of Linear Unidimensional Scaling (LUS) in Chapter 4, it is possible to find and fit (more general) anti-Robinson (AR) forms to two-mode proximity matrices. The same type of reordering strategy implemented in Section 4.1 by ordertm.m would be used, but the more general AR form would be fitted to the reordered square proximity matrix, $P_{p_0}^{(tm)} = \{p_{p_0(i)p_0(j)}^{(tm)}\}$; the least-
squares criterion
\[
\sum_{i,j=1}^{n} w_{\rho_0(i)\rho_0(j)}(P_{\rho_0(i)\rho_0(j)}^{(tm)} - \hat{p}_{ij})^2,
\]
is minimized, where \(w_{\rho_0(i)\rho_0(j)} = 0\) if \(\rho_0(i)\) and \(\rho_0(j)\) are both row or both column objects, and = 1 otherwise. The entries in the matrix \(\{\hat{p}_{ij}\}\) fitted to \(P_{\rho_0}^{(tm)}\) are AR in form (and which correspond to nonzero values of the weight function \(w_{\rho_0(i)\rho_0(j)}\)), and thus satisfy certain linear inequality constraints generated from how the row and column objects are intermixed by the given permutation \(\rho_0(\cdot)\). We note here and discuss this more completely in the section to follow that the patterning of entries in \(\{\hat{p}_{ij}\}\) fitted to the original two-mode proximity matrix, with appropriate row and column permutations extracted from \(\rho_0\), is called an anti-Q-form.

### 11.1 Fitting and Finding Two-Mode AR Matrices

The m-file `aroebfittm.m` does a confirmatory two-mode anti-Robinson fitting of a given ordering of the row and column objects of a two-mode proximity matrix using the Dykstra-Kaczmarz iterative projection least-squares method. The usage syntax has the form

```
[fit, vaf, rowperm, colperm] = aroebfittm(proxtm, inperm)
```

where `PROXTM` is the input two-mode proximity matrix; `INPERM` is the given ordering of the row and column objects together; `FIT` is an \(n_a \times n_b\) (number of rows by number of columns) matrix fitted to `PROXTM(ROWPERM, COLPERM)` with `VAF` being the variance-accounted-for based on the (least-squares criterion) sum of squared discrepancies between `PROXTM(ROWPERM, COLMEAN)` and `FIT`; `ROWPERM` and `COLPERM` are the row and column object orderings derived from `INPERM`.

The matrix given by `FIT` that is intended to approximate the row and column permuted two-mode proximity matrix, `PROXTM(ROWPERM, COLPERM)`, displays a particularly important patterning of its entries called an anti-Q-form in the literature (see Hubert and Arabie, 1995a, for an extended discussion of this type of patterning for a two-mode matrix). Specifically, a matrix is said to have the anti-Q-form (for rows and columns) if within each row and column the entries are nonincreasing to a minimum and thereafter nondecreasing. Matrices satisfying the anti-Q-form have a convenient interpretation presuming an underlying unidimensional scale that jointly represents both the row and column objects. Explicitly, suppose a matrix has been appropriately row-ordered to display the anti-Q-form for columns. Any dichotomization of the entries within a column at some threshold value (using 0 for entries below the threshold and 1 if at or above), produces a matrix that has the consecutive zeros property within each column, that is, all zeros within a column occur consecutively, uninterrupted by intervening ones. In turn, any matrix with the consecutive zeros property for columns suggests the existence of a perfect scale (error-free), where row objects can be ordered along a continuum (using the same row order for the matrix that actually reflects the anti-Q-form for columns), and each column object is representable as an interval along the continuum (encompassing those consecutive row objects corresponding to zeros). Historically, the type of
11.1. Fitting and Finding Two-Mode AR Matrices

Pattern represented by the anti-Q-form has played a major role in the literature of (unidimensional) unfolding, and for example, is the basis of Coombs's (1964, Chapter 4) parallelogram structure for a two-mode proximity matrix. The reader is referred to Hubert (1974) for a review of some of these connections.

To provide an example of what an anti-Q-form looks like for our two-mode data matrix, goldfish_receptor, we will use arobfndtm.m both to find and fit an anti-Robinson form using iterative projection to a two-mode proximity matrix in the $L_2$-norm based on a permutation identified through the use of iterative quadratic assignment. The usage syntax is

$$[\text{fit}, \text{vaf}, \text{outperm}, \text{rowperm}, \text{colperm}] = \text{aro bfndtm(proxtm, inperm, kblock)}$$

where, again, INPERM is a given starting permutation of the first $n = n_a + n_b$ integers; FIT is the least-squares optimal matrix (with variance-accounted-for of VAF) displaying an anti-Q-form (because of the anti-Robinson form constructed for the combined row and column object ordering given by the ending permutation OUTPERM). KBLOCK defines the block size in the use the iterative quadratic assignment routine. ROWPERM and COLPERM are the resulting row and column permutations for the objects. In the listing below, the VAF for the given fitted matrix is very high: .9667 (which can be compared to the alternative representations given earlier with values of .8072 (linear unidimensional scaling), .6209 (ultrametric), and .8663 (additive tree)).

```matlab
>> load goldfish_receptor.dat
>> [fit, vaf, outperm, rowperm, colperm] = ...
    arobfndtm(proxtm, inperm, kblock)
```

In the listing below, the VAF for the given fitted matrix is very high: .9667 (which can be compared to the alternative representations given earlier with values of .8072 (linear unidimensional scaling), .6209 (ultrametric), and .8663 (additive tree)).

```matlab
>> load goldfish_receptor.dat
>> [fit, vaf, outperm, rowperm, colperm] = ... 
    arobfndtm(goldfish_receptor, randperm(20), 2);
>> fit

Columns 1 through 6

```

```
68.0000  54.5000  80.0000  138.0000  145.0000  162.8000
71.5000  54.5000  64.0000  128.0000  144.0000  162.8000
71.5000  47.0000  61.0000  117.5000  117.5000  145.0000
80.0000  47.5000  47.5000  98.0000  116.0000  137.5000
155.0000 108.0000  63.0000  94.0000  103.0000  137.5000
174.0000 125.0000  84.0000  49.0000  47.6667  76.0000
200.0000 143.0000  91.0000  49.0000  47.6667  76.0000
200.0000 156.0000 107.0000  67.0000  47.6667  60.0000
200.0000 183.0000 177.0000 176.0000 168.0000 112.5000
200.0000 200.0000 200.0000 198.0000 186.0000 112.5000
200.0000 200.0000 200.0000 198.0000 188.0000 143.0000
```

Columns 7 through 9
162.8000 200.0000 200.0000  
162.8000 162.8000 173.0000  
145.0000 151.6667 158.0000  
138.5000 151.6667 158.0000  
138.5000 151.6667 158.0000  
106.0000 134.5000 134.5000  
106.0000 124.5000 124.5000  
78.0000 100.0000 100.0000  
82.5000  47.0000  46.0000  
82.5000  54.0000  47.5000  
111.0000  54.0000  47.5000  

>> vaf

vaf =

    0.9667

>> outperm

outperm =

Columns 1 through 10

   20   11   10   19    9    8    7   17   16

Columns 11 through 20

    6    5    4   15   14   13    3   12    2    1

>> rowperm'

ans =

Columns 1 through 10

   11   10    9    8    7    6    5    4    3    2

Column 11

    1

>> colperm'

ans =
11.2 Multiple Two-Mode AR Reorderings and Fittings

The m-file, *biarobfndtm.m*, finds and fits the sum of two anti-Q-forms (extracted from fitting two anti-Robinson matrices) using iterative projection to a two-mode proximity matrix in the $L_2$-norm based on permutations identified through the use of iterative quadratic assignment. In the usage

\[
\begin{align*}
\text{[find, vaf, targone, targtwo, outpermone, outpermtwo, ...} \\
\text{rowpermone, colpermone, rowpermtwo, colpermtwo] = ...}
\end{align*}
\]

*PROXTM* is the usual input two-mode proximity matrix ($n_a \times n_b$) with a dissimilarity interpretation, and *FIND* is the least-squares optimal matrix (with variance-accounted-for of VAF) to *PROXTM*. The latter matrix *PROXTM* is the sum of the two matrices *TARGONE* and *TARGTWO* based on the two row and column object orderings given by the ending permutations *OUTPERMONE* and *OUTPERMTWO*. The two ending permutations of *OUTPERMONE* and *OUTPERMTWO* contain the ending row and column object orderings of *ROWPERMONE* and *ROWPERMTWO* and *COLPERMONE* and *COLPERMTWO*. *KBLOCK* defines the block size in the use the iterative quadratic assignment routine; the input permutations are *INPERMONE* and *INPERMTWO*.

As can be seen in the example below, the sum of two anti-Q-forms fitted to the *goldfish_receptor* data provides an almost perfect reconstruction (with a VAF of .9991).

\[
\begin{align*}
\text{[find, vaf, targone, targtwo, outpermone, outpermtwo, ...} \\
\text{rowpermone, colpermone, rowpermtwo, colpermtwo] = ...}
\end{align*}
\]

\[
\text{>> biarobfndtm(goldfish_receptor, randperm(20), randperm(20), 2);}
\]

\[
\text{>> find}
\]

Columns 1 through 6

\[
\begin{align*}
46.8209 &\quad 52.8209 &\quad 111.0000 &\quad 143.0179 &\quad 187.7706 &\quad 196.0000 \\
47.4432 &\quad 55.0805 &\quad 75.0000 &\quad 100.0000 &\quad 186.0805 &\quad 200.1519 \\
46.4432 &\quad 46.3665 &\quad 90.0000 &\quad 125.0000 &\quad 167.8209 &\quad 175.5000 \\
99.8209 &\quad 99.8209 &\quad 78.0000 &\quad 59.9942 &\quad 46.0000 &\quad 67.0000 \\
124.3209 &\quad 124.3209 &\quad 115.0000 &\quad 79.0058 &\quad 47.9762 &\quad 46.5000 \\
115.0000 &\quad 153.4366 &\quad 97.0000 &\quad 73.0000 &\quad 48.0805 &\quad 52.1519 \\
198.1887 &\quad 182.8894 &\quad 154.9740 &\quad 148.9740 &\quad 103.0000 &\quad 94.0000 \\
135.0000 &\quad 151.0826 &\quad 123.0000 &\quad 127.0000 &\quad 116.0000 &\quad 98.2324 \\
141.0000 &\quad 113.0000 &\quad 142.0000 &\quad 148.0000 &\quad 114.4587 &\quad 120.4638 \\
173.8758 &\quad 151.1443 &\quad 176.5000 &\quad 176.5000 &\quad 143.8209 &\quad 127.8209
\end{align*}
\]
199.8209 199.8209 160.0000 160.9821 144.7706 138.0000

Columns 7 through 9

200.0000 199.8209 199.8209
200.0805 200.0805 200.5568
176.9762 182.8209 199.5568
107.0000 155.8209 199.4801
90.8209 142.8209 199.4801
84.0805 125.0805 174.0000
63.0000 108.0000 155.9740
49.0000 50.9174 80.0000
61.0000 47.0000 54.0000
63.8209 50.9791 88.1242
80.0000 53.0000 67.8209

>> vaf

vaf =

0.9991

>> targone

targone =

Columns 1 through 6

47.7330 53.7330 111.0625 142.8045 189.4628 196.5815
47.2786 50.8978 84.2250 116.6250 181.8978 195.8390
47.2786 47.2786 84.2250 116.6250 168.7330 177.8883
100.7330 100.7330 77.4028 54.9906 54.9906 67.5815
125.2330 125.2330 110.3021 74.0022 48.8883 48.8883
149.2540 149.2540 110.3021 82.1000 43.8978 47.8390
167.3557 152.0564 124.1410 118.1410 102.8889 93.9195
167.3557 152.0564 136.1125 136.1000 115.5496 93.9195
167.3557 152.0564 152.0000 148.3003 116.1509 116.1509
167.3557 152.0564 152.0564 152.0564 144.7330 128.7330
200.7330 200.7330 160.7688 160.7688 146.4628 138.0000

Columns 7 through 9

200.5815 200.7330 200.7330
195.8978 195.8978 200.3922
177.8883 183.7330 200.3922
107.5815 156.7330 200.3922
11.2. Multiple Two-Mode AR Reorderings and Fittings

\[
\begin{array}{ccc}
91.7330 & 143.7330 & 200.3922 \\
79.8978 & 120.8978 & 174.0833 \\
62.8889 & 107.2500 & 125.1410 \\
48.5496 & 51.8912 & 81.6042 \\
61.5815 & 51.8912 & 81.6042 \\
64.7330 & 51.8912 & 81.6042 \\
80.5815 & 54.5000 & 68.7330
\end{array}
\]

>> targtwo

targtwo =

Columns 1 through 6

\[
\begin{array}{cccccccc}
-16.6250 & -9.2250 & 0.1646 & 0.1646 & 4.1826 & 4.1826 \\
-9.1000 & -13.1125 & -32.3557 & -1.6042 & -0.9738 & -0.9738 \\
-0.3003 & -10.0000 & -26.3557 & -27.6042 & -39.0564 & -4.8912 \\
0.2134 & -0.7688 & -0.9121 & -0.9121 & -0.9121 & -1.5000 \\
0.2134 & -0.0625 & -0.9121 & -0.9121 & -0.9121 & -0.9121 \\
5.0036 & 0.5972 & -0.9121 & -0.9121 & -0.9121 & -0.9121 \\
5.0036 & 4.6979 & -0.9121 & -0.9121 & -0.9121 & -0.9121 \\
8.3750 & 5.7750 & -0.8354 & -0.8354 & -0.9121 & -0.9121 \\
24.4436 & 24.4436 & 6.5200 & 6.5200 & -0.9121 & -0.9121 \\
30.8330 & 30.8330 & 30.8330 & 30.8330 & 30.8330 & 0.7500
\end{array}
\]

Columns 7 through 9

\[
\begin{array}{cccc}
4.1826 & 4.1826 & 4.3129 \\
4.1826 & 4.1826 & 4.3129 \\
0.4504 & 0.4504 & 4.3129 \\
-1.6922 & -0.5815 & 4.3129 \\
-1.6922 & -0.5815 & 0 \\
-1.6922 & -0.5815 & -0.5815 \\
-8.9906 & -0.5815 & -0.5815 \\
-0.9121 & -0.9121 & -2.3883 \\
-0.9121 & -0.9121 & -2.3883 \\
-0.9121 & -0.9121 & -0.9121 \\
0.1111 & 0.1111 & 0.0805
\end{array}
\]

>> outpermone

outpermone =

Columns 1 through 10
Chapter 11. Anti-Robinson (AR) Matrices for Two-Mode Proximity Data

\[
\begin{array}{cccccccccc}
1 & 12 & 2 & 13 & 3 & 14 & 4 & 15 & 5 & 16 \\
\end{array}
\]

Columns 11 through 20

\[
\begin{array}{cccccccccc}
6 & 17 & 7 & 8 & 18 & 9 & 10 & 19 & 11 & 20 \\
\end{array}
\]

>> outpermtwo

\[
\text{outpermtwo} =
\begin{array}{cccccccccc}
\end{array}
\]

Columns 1 through 10

\[
\begin{array}{cccccccccc}
15 & 2 & 14 & 6 & 12 & 8 & 20 & 9 & 13 & 19 \\
\end{array}
\]

Columns 11 through 20

\[
\begin{array}{cccccccccc}
11 & 1 & 4 & 16 & 18 & 5 & 3 & 17 & 10 & 7 \\
\end{array}
\]

>> rowpermone'

\[
\text{ans} =
\begin{array}{cccccccccc}
\end{array}
\]

Columns 1 through 10

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]

Column 11

11

>> colpermone'

\[
\text{ans} =
\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\]

>> rowpermtwo'

\[
\text{ans} =
\begin{array}{cccccccccc}
\end{array}
\]

Columns 1 through 10

\[
\begin{array}{cccccccccc}
2 & 6 & 8 & 9 & 11 & 1 & 4 & 5 & 3 & 10 \\
\end{array}
\]

Column 11
11.2. Multiple Two-Mode AR Reorderings and Fittings

7

>> colpermtwo'

ans =

4  3  1  9  2  8  5  7  6
Chapter 11. Anti-Robinson (AR) Matrices for Two-Mode Proximity Data
Bibliography


Appendix A

Header comments for the m-files mentioned in alphabetical order

arobfit.m

function [fit, vaf] = arobfir(prox, inperm)

% AROBFIT fits an anti-Robinson matrix using iterative projection to
% a symmetric proximity matrix in the $L_{2}$-norm.
% PROX is the input proximity matrix ($n \times n$ with a zero main
Appendix A. Header comments for the mentioned m-files

% diagonal and a dissimilarity interpretation);
% INPERM is a given permutation of the first $n$ integers;
% FIT is the least-squares optimal matrix (with variance-
% accounted-for of VAF) to PROX having an anti-Robinson form for
% the row and column object ordering given by INPERM.

**arobfittm.m**

function [fit,vaf,rowperm,colperm] = arobfittm(proxtm,inperm)

% AROBFITTM does a confirmatory two-mode anti-Robinson fitting of a
given ordering of the row and column objects of a two-mode
proximity matrix PROXTM using Dykstra's (Kaczmarz's)
iterative projection least-squares method.
% INPERM is the given ordering of the row and column objects
together; FIT is an nrow (number of rows) by ncol (number of
columns) matrix fitted to PROXTM(ROWPERM,COLPERM)
with VAF being the variance-accounted for and
based on the (least-squares criterion) sum of
squared discrepancies between FIT and PROXTM(ROWPERM,COLMEAN);
ROWPERM and COLPERM are the row and column object orderings
derived from INPERM.

**arobfnd.m**

function [find, vaf, outperm] = arobfnd(prox, inperm, kblock)

% AROBFND finds and fits an anti-Robinson
% matrix using iterative projection to
% a symmetric proximity matrix in the $L_2$-norm based on a
% permutation identified through the use of iterative quadratic
% assignment.
% PROX is the input proximity matrix ($n \times n$ with a zero main
diagonal and a dissimilarity interpretation);
% INPERM is a given starting permutation of the first $n$ integers;
% FIT is the least-squares optimal matrix (with
% variance-accounted-for of VAF) to PROX having an anti-Robinson
% form for the row and column object ordering given by the ending
% permutation OUTPERM. KBLOCK defines the block size in the use the
% iterative quadratic assignment
% routine.
arobfndtm.m

function [fit, vaf, outperm, rowperm, colperm] = ...
arobfndtm(proxtm, inperm, kblock)

% AROBFNDTM finds and fits an anti-Robinson
% form using iterative projection to
% a two-mode proximity matrix in the $L_{2}$-norm based on a
% permutation identified through the use of iterative quadratic
% assignment.
% PROXTM is the input two-mode proximity matrix
% ($n_{a} \times n_{b}$) with a zero main diagonal
% and a dissimilarity interpretation);
% INPERM is a given starting permutation
% of the first $n = n_{a} \times n_{b}$ integers;
% FIT is the least-squares optimal matrix (with
% variance-accounted-for of VAF) to PROXTM having the anti-Robinson
% form for the row and column
% object ordering given by the ending permutation OUTPERM. KBLOCK
% defines the block size in the use the iterative quadratic
% assignment routine. ROWPERM and COLPERM are the resulting
% row and column permutations for the objects.

atreectul.m

function [find, vaf] = atreectul(prox, inperm)

% ATREECTUL finds and fits an additive tree by first fitting
% a centroid metric (using centfit.m) and
% secondly an ultrametric to the residual
% matrix (using ultrafnd.m).
% PROX is the input proximity matrix (with a zero main diagonal
% and a dissimilarity interpretation);
% INPERM is a permutation that determines the order in which the
% inequality constraints are considered;
% FIND is the found least-squares matrix (with variance-accounted-
% for of VAF) to PROX satisfying the additive tree constraints.

atreeedec.m

function [ulmetric, ctmetric] = atreeedec(prox, constant)

% ATREEDEC decomposes a given additive tree matrix into an
% ultrametric and a centroid metric matrix (where the root is
% half-way along the longest path).
% PROX is the input proximity matrix (with a zero main diagonal
% and a dissimilarity interpretation);
% CONSTANT is a non-negative number (less than or equal to the
% maximum proximity value) that controls the
% positivity of the constructed ultrametric values;
% ULMETRIC is the ultrametric component of the decomposition;
% CTMETRIC is the centroid metric component of the decomposition
% (given by values $g_{1},...,g_{n}$ for each of the objects,
% some of which may actually be negative depending on the input
% proximity matrix used).

atreefit.m

function [fit,vaf] = atreefit(prox,targ)

% ATREEFIT fits a given additive tree using iterative projection to
% a symmetric proximity matrix in the $L_{2}$-norm.
% PROX is the input proximity matrix (with a zero main diagonal
% and a dissimilarity interpretation);
% TARG is an matrix of the same size as PROX with entries
% satisfying the four-point additive tree constraints;
% FIT is the least-squares optimal matrix (with
% variance-accounted-for of VAF) to PROX satisfying the
% additive tree constraints implicit in TARG.

atreefnd.m

function [find,vaf] = atreefnd(prox,inperm)

% ATREEFND finds and fits an additive tree using iterative projection
% heuristically on a symmetric proximity matrix in the $L_{2}$-norm.
% PROX is the input proximity matrix (with a zero main diagonal
% and a dissimilarity interpretation);
% INPERM is a permutation that determines the order in which the
% inequality constraints are considered;
% FIND is the found least-squares matrix (with variance-accounted-for
% of VAF) to PROX satisfying the additive tree constraints.

atreefndtm.m

function [find,vaf,ultrafit,lengths] = ...
atreefndtm(proxtm,inpermrow,inpermcol)

% ATREEFNDTM finds and fits a two-mode additive tree;  
% iterative projection is used  
% heuristically to find a two-mode ultrametric component that  
% is added to a two-mode centroid metric to  
% produce the two-mode additive tree.  
% PROXTM is the input proximity matrix  
% (with a dissimilarity interpretation);  
% INPERMROW and INPERMCOL are permutations for the row and column  
% objects that determine the order in which the  
% inequality constraints are considered;  
% FIND is the found least-squares matrix (with variance-accounted-for  
% of VAF) to PROXTM satisfying the additive tree constraints;  
% the vector LENGTHS contains the row followed by column values for  
% the two-mode centroid metric component;  
% ULTRAFIT is the ultrametric component.

biarobfnd.m

function [find,vaf,targone,targtwo,outpermone,outpermtwo] = ...  
biarobfnd(prox,inperm,kblock)

% BIAROBFND finds and fits the sum of two  
% anti-Robinson matrices using iterative projection to  
% a symmetric proximity matrix in the $L_{2}$-norm based on  
% permutations identified through  
% the use of iterative quadratic assignment.  
% PROX is the input proximity matrix ($n \times n$ with a zero  
% main diagonal and a dissimilarity interpretation);  
% INPERM is a given starting permutation of the first $n$ integers;  
% FIND is the least-squares optimal matrix (with  
% variance-accounted-for of VAF)  
% to PROX and is the sum of the two anti-Robinson matrices  
% TARGONE and TARGETWO based on the two row and column  
% object orderings given by the ending permutations OUTPERMONE  
% and OUTPERMTWO. KBLOCK defines the block size in the use the  
% iterative quadratic assignment routine.

biarobfndtm.m

function [find,vaf,targone,targtwo,outpermone,outpermtwo, ...  
rowpermone,colpermone,rowpermtwo,colpermtwo] = ...
Appendix A. Header comments for the mentioned m-files

biarobfndtm(proxtm,inpermone,inpermtwo,kblock)

% BIAROBFNDTM finds and fits the sum of
% two anti-Robinson matrices using iterative projection to
% a two-mode proximity matrix in the $L_{2}$-norm based on
% permutations identified through the use of
% iterative quadratic assignment.
% PROXTM is the input two-mode proximity matrix ($nrow \times ncol$)
% and a dissimilarity interpretation);
% FIND is the least-squares optimal matrix (with variance-
% accounted-for of VAF) to PROXTM and is the sum of the two matrices
% TARGONE and TARGTWO based on the two row and column
% object orderings given by the ending permutations OUTPERMONE
% and OUTPERMTWO, and in turn ROWPERMONE and ROWPERMTWO and
% COLPERMONE and COLPERMTWO. KBLOCK defines the block size
% in the use the iterative quadratic assignment routine;
% the input permutations are INPERMONE and
% INPERMTWO.

biatreefnd.m

function [find,vaf,targone,targtwo] = biatreefnd(prox,inperm)

% BIATREEFND finds and fits the sum
% of two additive trees using iterative projection
% heuristically on a symmetric proximity matrix in the $L_{2}$-norm.
% PROX is the input proximity matrix (with a zero main diagonal
% and a dissimilarity interpretation);
% INPERM is a permutation that determines the order in which the
% inequality constraints are considered;
% FIND is the found least-squares matrix (with variance-accounted-for
% of VAF) to PROX and is the sum of
% the two additive tree matrices TARGONE and TARGTWO.

bicirac.m

function [find,vaf,targone,targtwo,outpermiteone,outpermtwotwo,...
addconone,addcontwo] = bicirac(prox,inperm,kblock)

% BICIRAC finds and fits the sum of two circular
% unidimensional scales using iterative projection to
% a symmetric proximity matrix in the $L_{2}$-norm based on
% permutations identified through the use
bicirarobfnd.m

function [find,vaf,targone,targtwo,outpermone,outpermtwo] = ...
bicirarobfnd(prox,inperm,kblock)

bicirsarobfnd.m

function [find,vaf,targone,targtwo,outpermone,outpermtwo] = ...
bicirsarobfnd(prox,inperm,kblock)


\% diagonal and a dissimilarity interpretation);
\% INPERM is a given starting permutation of the first $n$ integers;
\% FIND is the least-squares optimal matrix (with variance-
\% accounted-for of VAF) to PROX and is the
\% sum of the two strongly circular-anti-Robinson matrices;
\% TARGONE and TARGTWO are based on the two row and column
\% object orderings given by the ending permutations OUTPERMONE
\% and OUTPERMTWO. KBLOCK defines the block size in the use the
\% iterative quadratic assignment routine.

bimonscalqa.m

function [outpermone,outpermtwo,coordone,coordtwo, ...
fitone,fittwo,addconone,addcontwo,vaf,monprox] = ...
bimonscalqa(prox,targone,targtwo,inpermone,inpermtwo,kblock,nopt)

\% BIMONCALQA carries out a bidimensional
\% scaling of a symmetric proximity
\% matrix using iterative quadratic assignment, plus it provides an
\% optimal monotonic transformation (MONPROX) of the original input
\% proximity matrix.
\% PROX is the input proximity matrix (with a zero main diagonal
\% and a dissimilarity interpretation);
\% TARGONE is the input target matrix for the
\% first dimension (usually with
\% a zero main diagonal and with a
\% dissimilarity interpretation representing
\% equally-spaced locations along a continuum);
\% TARGTWO is the input target
\% matrix for the second dimension;
\% INPERMONE is the input beginning permutation for the first
\% dimension (a permutation of the first $n$ integers);
\% INPERMTWO is the input beginning
\% permutation for the second dimension;
\% the insertion and rotation routines use from 1 to KBLOCK
\% (which is less than or equal to $n-1$) consecutive objects in
\% the permutation defining the row and column orders of the data
\% matrix; NOPT controls the confirmatory or exploratory fitting of
\% the unidimensional scales; a value of NOPT = 0 will fit in a
\% confirmatory manner the two scales indicated by INPERMONE
\% and INPERMTWO; a value of NOPT = 1 uses iterative QA
\% to locate the better permutations to fit;
\% OUTPERMONE is the final object permutation for the first
\% dimension; OUTPERMTWO is the final object
% permutation for the second dimension;
% COORDONE is the set of first dimension coordinates
% in ascending order; COORDTWO is the set of second
% dimension coordinates in ascending order;
% ADDCONONE is the additive constant for the first dimensional
% model; ADDCONTWO is the additive constant for the second
% dimensional model; VAF is the variance-accounted-for
% in MONPROX by the bidimensional scaling.

bimonscaltmac.m

function [find, vaf, targone, targtwo, outpermone, outpermtwo, ...
        rowpermone, colpermone, rowpermtwo, colpermtwo, addconone, ...
        addcontwo, coordone, coordtwo, axes, monproxtm] = ...
    bimonscaltmac(proxtm, inpermone, inpermtwo, kblock, nopt)

% BIMONSCLTMAC finds and fits the sum of two linear unidimensional
% scales using iterative projection to
% a two-mode proximity matrix in the $L_{2}$-norm based on
% permutations identified through the use of iterative quadratic
% assignment. It also provides an optimal monotonic transformation
% (MONPROX) of the original input proximity matrix.
% PROXTM is the input two-mode proximity matrix ($nrow \times ncol$
% with a dissimilarity interpretation);
% FIND is the least-squares optimal matrix (with variance-
% accounted-for of VAF) to the monotonic transformation MONPROXTM of
% the input proximity matrix and is the sum of the two matrices
% TARGONE and TARGTWO based on the two row and column
% object orderings given by the ending permutations OUTPERMONE
% and OUTPERMTWO, and in turn ROWPERMONE and ROWPERMTWO and
% COLPERMONE and COLPERMTWO. KBLOCK defines the block size in
% the use of the
% iterative quadratic assignment routine and ADDCONONE and ADDCONTWO
% are the two additive constants for the two model components; The
% $n$ coordinates are in COORDONE and COORDTWO. The input
% permutations are INPERMONE and INPERMTWO. The $n \times \times 2$
% matrix AXES gives the plotting coordinates for the
% combined row and column object set.
% NOPT controls the confirmatory or exploratory fitting of
% the unidimensional scales; a value of NOPT = 0 will fit in a
% confirmatory manner the two scales
% indicated by INPERMONE and INPERMTWO;
% a value of NOPT = 1 uses iterative QA
% to locate the better permutations to fit.
biplottm.m

function [] = biplottm(axes,nrow,ncol)

% BIPLOTTM plots the combined row and column object set using
% coordinates given in the $n \times 2$ matrix AXES; here the
% number of rows is NROW and the number of columns is NCOL,
% and $n$ is the sum of NROW and NCOL.
% The first NROW rows of AXES give the row object coordinates;
% the last NCOL rows of AXES give the column object coordinates.
% The plotting symbol for rows is a circle (o);
% for columns it is an asterisk (*).
% The labels for rows are from 1 to NROW;
% those for columns are from 1 to NCOL.

bisarobfnd.m

function [find,vaf,targone,targtwo,outpermone,outpermtwo] = 
    bisarobfnd(prox,inperm,kblock)

% BISAROBFND finds and fits the sum of two
% strongly anti-Robinson matrices using iterative
% projection to a symmetric proximity matrix in
% the $L_2$-norm based on permutations
% identified through the use of iterative quadratic assignment.
% PROX is the input proximity matrix ($n \times n$ with a zero
% main diagonal and a dissimilarity interpretation);
% INPERM is a given starting permutation of the first $n$ integers;
% FIND is the least-squares optimal matrix (with
% variance-accounted-for of VAF) to PROX and is the sum of the two
% strongly anti-Robinson matrices;
% TARGONE and TARGTWO are based on the two row and column
% object orderings given by the ending permutations OUTPERMONE
% and OUTPERMTWO. KBLOCK defines the block size in the use the
% iterative quadratic assignment routine.

biscalqa.m

function [outpermone,outpermtwo,coordone,coordtwo,...
    fitone,fittwo,addconone,addcontwo,vaf] = ...
    biscalqa(prox,targone,targtwo,inpermone,inpermtwo,kblock,nopt)
% BISCALQA carries out a bidimensional scaling of a symmetric
% proximity matrix using iterative quadratic assignment.
% PROX is the input proximity matrix (with a zero main diagonal
% and a dissimilarity interpretation);
% TARGONE is the input target matrix for the first dimension
% (usually with a zero main diagonal and a dissimilarity
% interpretation representing equally-spaced locations along
% a continuum); TARGTWO is the input target
% matrix for the second dimension;
% INPERMONE is the input beginning permutation for the first
% dimension (a permutation of the first $n$ integers);
% INPERMTWO is the input beginning
% permutation for the second dimension;
% the insertion and rotation routines use from 1 to KBLOCK
% (which is less than or equal to $n-1$) consecutive objects in
% the permutation defining the row and column orders of the data
% matrix. NOPT controls the confirmatory or exploratory fitting
% of the unidimensional scales; a value of NOPT = 0 will fit in a
% confirmatory manner the two scales
% indicated by INPERMONE and INPERMTWO;
% a value of NOPT = 1 uses iterative QA
% to locate the better permutations to fit;
% OUTPERMONE is the final object permutation for the
% first dimension; OUTPERMTWO is the final object permutation
% for the second dimension;
% COORDONE is the set of first dimension coordinates
% in ascending order; COORDTWO is the set of second dimension
% coordinates in ascending order;
% ADDCONONE is the additive constant for the first
% dimensional model; ADDCONTWO is the additive constant for
% the second dimensional model;
% VAF is the variance-accounted-for in PROX by
% the bidimensional scaling.

biscaltmac.m

function [find,vaf,targone,targtwo,outpermone,outpermtwo, ...
rowpermone,colpermone,rowpermtwo,colpermtwo,addconone,...
addcontwo,coordone,coordtwo,axes] = ...
biscaltmac(proxtm,inpermone,inpermtwo,kblock,nopt)

% BISCALTMAC finds and fits the sum of two linear
% unidimensional scales using iterative projection to % a two-mode proximity matrix in the $L_{2}$-norm based on % permutations identified through the use of iterative quadratic % assignment.
% PROXTM is the input two-mode proximity matrix ($nrow \times ncol$ % with a dissimilarity interpretation);
% FIND is the least-squares optimal matrix (with variance-accounted-% for of VAF) to PROXTM and is the sum of the two matrices % TARGONE and TARGTWO based on the two row and column % object orderings given by the ending permutations OUTPERMONE % and OUTPERMTWO, and in turn ROWPERMONE and ROWPERMTWO and % COLPERMONE and COLPERMTWO. KBLOCK defines the block size % in the use the iterative quadratic assignment routine and % ADDCONONE and ADDCONTWO are % the two additive constants for the two model components;
% The $n$ coordinates % are in COORDONE and COORDTWO. The input permutations are INPERMONE % and INPERMTWO. The $n \times 2$ matrix AXES gives the % combined row and column object set.
% NOPT controls the confirmatory or % exploratory fitting of the unidimensional % scales; a value of NOPT = 0 will % fit in a confirmatory manner the two scales % indicated by INPERMONE and INPERMTWO;
% a value of NOPT = 1 uses iterative QA % to locate the better permutations to fit.

**biultrafnd.m**

```matlab
function [find,vaf,targone,targtwo] = biultrafnd(prox,inperm)

% BIULTRAFND finds and fits the sum % of two ultrametrics using iterative projection % heuristically on a symmetric proximity matrix in the $L_{2}$-norm. % PROX is the input proximity matrix (with a zero main diagonal % and a dissimilarity interpretation);
% INPERM is a permutation that determines the order in which the % inequality constraints are considered;
% FIND is the found least-squares matrix (with variance-accounted-for % of VAF) to PROX and is the sum % of the two ultrametric matrices TARGONE and TARGTWO.
```
centfit.m

function [fit, vaf, lengths] = centfit(prox)

% CENTFIT finds the least-squares fitted centroid metric (FIT) to
% PROX, the input proximity matrix (with a zero main diagonal
% and a dissimilarity interpretation);
% The $n$ values that serve to define the approximating sums,
% $g_{i} + g_{j}$, are given in the vector LENGTHS of size $n \times 1$.

centfittm.m

function [fit, vaf, lengths] = centfittm(proxtm)

% CENTFITTM finds the least-squares fitted two-mode centroid metric
% (FIT) to PROXTM, the two-mode rectangular input proximity matrix
% (with a dissimilarity interpretation);
% The $n$ values (where $n$ = number of rows + number of columns)
% serve to define the approximating sums,
% $u_{i} + v_{j}$, where the $u_{i}$ are for the rows and the $v_{j}$
% are for the columns; these are given in the vector LENGTHS of size
% $n \times 1$, with row values first followed by the column values.

cirarobfit.m

function [fit, vaf] = cirarobfit(prox, inperm, targ)

% CIRAROBFIT fits a circular anti-Robinson matrix using iterative
% projection to a symmetric proximity matrix in the $L_2$-norm.
% PROX is the input proximity matrix ($n \times n$ with a zero
% main diagonal and a dissimilarity interpretation);
% INPERM is a given permutation of the first $n$ integers (around
% a circle); TARG is a given $n \times n$ matrix having the
% circular anti-Robinson form that guides the direction in which
% distances are taken around the circle.
% FIT is the least-squares optimal matrix (with variance-
% accounted-for of VAF) to PROX having an circular anti-Robinson
% form for the row and column object ordering given by INPERM.

cirarobfnd.m

function [find, vaf, outperm] = cirarobfnd(prox, inperm, kblock)
Appendix A. Header comments for the mentioned m-files

cirarobfnd_ac.m

function [fit, vaf, outperm] = cirarobfnd_ac(prox, inperm, kblock)

% CIRAROBFND finds and fits a circular anti-Robinson matrix using iterative projection to
% a symmetric proximity matrix in the $L_{2}$-norm based on a
% permutation identified through the use of iterative quadratic assignment.
% PROX is the input proximity matrix ($n \times n$ with a zero main diagonal and a dissimilarity interpretation);
% INPERM is a given starting permutation (assumed to be around the circle) of the first $n$ integers; FIT is the least-squares optimal matrix (with variance-accounted-for of VAF) to PROX having a circular anti-Robinson form for the row and column object ordering given by the ending permutation OUTPERM. KBLOCK defines the block size in the use the iterative quadratic assignment routine.

cirfit.m

function [fit, diff] = cirfit(prox, inperm)

% CIRFIT does a confirmatory fitting of a given order
% (assumed to reflect a circular ordering around a closed unidimensional structure) using Dykstra’s (Kaczmarz’s) iterative projection least-squares method.
% INPERM is the given order; FIT is an $n \times n$ matrix that is fitted to PROX(INPERM,INPERM) with least-squares value DIFF.
cirfitac.m

function [fit, vaf, addcon] = cirfitac(prox,inperm)

  % CIRFITAC does a confirmatory fitting (including
  % the estimation of an additive constant) for a given order
  % (assumed to reflect a circular ordering around a closed
  % unidimensional structure) using the Dykstra-
  % Kaczmarz iterative projection least-squares method.
  % INPERM is the given order; FIT is an $n \times n$ matrix that
  % is fitted to PROX(INPERM,INPERM) with variance-accounted-for of
  % VAF; ADDCON is the estimated additive constant.

cirfitac_ftarg.m

function [fit, vaf, addcon] = cirfitac_ftarg(prox,inperm,targ)

  % CIRFITAC_FTARG does a confirmatory fitting (including
  % the estimation of an additive constant) for a given order
  % (assumed to reflect a circular ordering around a closed
  % unidimensional structure) using the Dykstra-
  % Kaczmarz iterative projection least-squares method.
  % The inflection points are implicitly given by TARG which
  % is assumed to reflect a circular ordering of the same size as
  % PROX. INPERM is the given order; FIT is an $n \times n$ matrix
  % that is fitted to PROX(INPERM,INPERM) with variance-
  % accounted-for of VAF; ADDCON is the estimated additive constant.

cisarobfit.m

function [fit, vaf] = cisarobfit(prox,inperm,target)

  % CIRSAROBFIT fits a strongly circular anti-Robinson matrix
  % using iterative projection to
  % a symmetric proximity matrix in the $L_{2}$-norm.
  % PROX is the input proximity matrix ($n \times n$ with a zero
  % main diagonal and a dissimilarity interpretation);
  % INPERM is a given permutation of the first $n$ integers
  % (around a circle);
  % TARGET is a given $n \times n$ matrix having the circular
  % anti-Robinson form that guides the direction in which distances
% are taken around the circle.
% FIT is the least-squares optimal matrix (with variance-
% accounted-for of VAF) to PROX having a strongly circular
% anti-Robinson form for the row and column object ordering
% given by INPERM.

**cirsarobfnd.m**

function [find, vaf, outperm] = cirsarobfnd(prox, inperm, kblock)

% CIRSAROBFDN finds and fits a strongly circular
% anti-Robinson matrix using iterative projection to
% a symmetric proximity matrix in the $L_{2}$-norm based on a
% permutation identified through the use of
% iterative quadratic assignment.
% PROX is the input proximity matrix ($n \times n$ with a zero
% main diagonal and a dissimilarity interpretation);
% INPERM is a given starting permutation (assumed to be around the
% circle) of the first $n$ integers;
% FIT is the least-squares optimal matrix (with variance-
% accounted-for of VAF) to PROX having a strongly
% circular anti-Robinson form for the row and column
% object ordering given by the ending permutation OUTPERM. KBLOCK
% defines the block size in the use of the iterative
% quadratic assignment routine.

**cirsarobfnd_ac.m**

function [fit, vaf, outperm] = cirsarobfnd_ac(prox, inperm, kblock)

% CIRSAROBFDN fits a strongly circular
% anti-Robinson matrix using iterative projection to
% a symmetric proximity matrix in the $L_{2}$-norm based on a
% permutation identified through the use of
% iterative quadratic assignment.
% PROX is the input proximity matrix ($n \times n$ with a zero
% main diagonal
% and a dissimilarity interpretation);
% INPERM is a given starting permutation (assumed to be around the
% circle) of the first $n$ integers;
% FIT is the least-squares optimal matrix (with variance-
% accounted-for of VAF) to PROX having a strongly
% circular anti-Robinson form for the row and column
insertqa.m

function [outperm, rawindex, allperms, index] = insertqa(prox, targ, inperm, kblock)

insertqa(prox, targ, inperm, kblock)

% INSERTQA carries out an iterative
% Quadratic Assignment maximization task using the
% insertion of from 1 to KBLOCK
% (which is less than or equal to $n-1$) consecutive objects in
% the permutation defining the row and column order of the data
% matrix.
% INPERM is the input beginning permutation
% (a permutation of the first $n$ integers).
% PROX is the $n \times n$ input proximity matrix.
% TARG is the $n \times n$ input target matrix.
% OUTPERM is the final permutation of PROX with the cross-product
% index RAWINDEX with respect to TARG.
% ALLPERMS is a cell array containing INDEX entries corresponding
% to all the permutations identified in the optimization from
% ALLPERMS{1} = INPERM to ALLPERMS{INDEX} = OUTPERM.

linfit.m

function [fit, diff, coord] = linfit(prox,inperm)

linfit(prox,inperm)

% LINFIT does a confirmatory fitting of a given
% unidimensional order using Dykstra’s
% (Kaczmarz’s) iterative projection least-squares method.
% INPERM is the given order;
% FIT is an $n \times n$ matrix that is fitted to
% PROX(INPERM,INPERM) with least-squares value DIFF;
% COORD gives the ordered coordinates whose absolute differences
% could be used to reconstruct FIT.

linfitac.m

function [fit, vaf, coord, addcon] = linfitac(prox,inperm)

linfitac(prox,inperm)
Appendix A. Header comments for the mentioned m-files


% LINFITAC does a confirmatory fitting of a given unidimensional order
% using the Dykstra-Kaczmarz iterative projection
% least-squares method, but differing from LINFIT.M in
% including the estimation of an additive constant.
% INPERM is the given order;
% FIT is an $n \times n$ matrix that is fitted to
% PROX(INPERM,INPERM) with variance-accounted-for VAF;
% COORD gives the ordered coordinates whose absolute differences
% could be used to reconstruct FIT; ADDCON is the estimated
% additive constant that can be interpreted as being added to PROX.

linfittm.m

function [fit,diff,rowperm,colperm,coord] = linfittm(proxtm,inperm)

% LINFITTM does a confirmatory two-mode fitting of a given
% unidimensional ordering of the row and column objects of
% a two-mode proximity matrix PROXTM using Dykstra's (Kaczmarz's)
% iterative projection least-squares method.
% INPERM is the given ordering of the row and column objects
% together; FIT is an nrow (number of rows) by ncol (number
% of columns) matrix of absolute coordinate differences that
% is fitted to PROXTM(ROWPERM,COLPERM) with DIFF being the
% (least-squares criterion) sum of squared discrepancies
% between FIT and PROXTM(ROWPERM,COLMEAN);
% ROWPERM and COLPERM are the row and column object orderings
% derived from INPERM. The nrow + ncol coordinates
% (ordered with the smallest
% set at a value of zero) are given in COORD.

linfittmac.m

function [fit,vaf,rowperm,colperm,addcon,coord] = ...
linfittmac(proxtm,inperm)

% LINFITTMAC does a confirmatory two-mode fitting of a given
% unidimensional ordering of the row and column objects of
% a two-mode proximity matrix PROXTM using Dykstra's (Kaczmarz's)
% iterative projection least-squares method;
% it differs from LINFITTM.M by including the estimation of an
% additive constant.
% INPERM is the given ordering of the row and column objects
% together; FIT is an nrow (number of rows) by ncol (number
% of columns) matrix of absolute coordinate differences that
% is fitted to PROXTM(ROWPERM, COLPERM) with VAF being the
% variance-accounted-for. ROWPERM and COLPERM are the row and
% column object orderings derived from INPERM. ADDCON is the
% estimated additive constant that can be interpreted as being
% added to PROXTM (or alternatively subtracted
% from the fitted matrix FIT). The nrow + ncol coordinates
% (ordered with the smallest
% set at a value of zero) are given in COORD.

order.m

function [outperm, rawindex, allperms, index] = ...
order(prox, targ, inperm, kblock)

% ORDER carries out an iterative Quadratic Assignment maximization
% task using a given square ($n \times n$) proximity matrix PROX (with
% a zero main diagonal and a dissimilarity interpretation).
% Three separate local operations are used to permute
% the rows and columns of the proximity matrix to maximize the
% cross-product index with respect to a given square target matrix
% TARG: pairwise interchanges of objects in the permutation defining
% the row and column order of the square proximity matrix;
% the insertion of from 1 to KBLOCK
% (which is less than or equal to $n-1$) consecutive objects in
% the permutation defining the row and column order of the data
% matrix; the rotation of from 2 to KBLOCK
% (which is less than or equal to $n-1$) consecutive objects in
% the permutation defining the row and column order of the data
% matrix. INPERM is the input beginning permutation (a permutation
% of the first $n$ integers).
% OUTPERM is the final permutation of PROX with the
% cross-product index RAWINDEX
% with respect to TARG. ALLPERMS is a cell array containing INDEX
% entries corresponding to all the
% permutations identified in the optimization from ALLPERMS{1} =
% INPERM to ALLPERMS{INDEX} = OUTPERM.

ordertm.m

function [outperm, rawindex, allperms, index, squareprox] = ...
ordertm(proxtm, targ, inperm, kblock)
% ORDERTM carries out an iterative
% Quadratic Assignment maximization task using the
% two-mode proximity matrix PROXTM
% (with entries deviated from the mean proximity)
% in the upper-right- and lower-left-hand portions of
% a defined square ($n \times n$) proximity matrix
% (called SQUAREPROX with a dissimilarity interpretation)
% with zeros placed elsewhere ($n = \text{number of rows} +
% \text{number of columns of PROXTM = nrow + ncol}$);
% three separate local operations are used to permute
% the rows and columns of the square
% proximity matrix to maximize the cross-product
% index with respect to a square target matrix TARG:
% pairwise interchanges of objects in the
% permutation defining the row and column
% order of the square proximity matrix; the insertion of from 1 to
% KBLOCK (which is less than or equal to $n-1$) consecutive objects
% in the permutation defining the row and column order of the
% data matrix; the rotation of from 2 to KBLOCK (which is less than
% or equal to $n-1$) consecutive objects in
% the permutation defining the row and column order of the data
% matrix. INPERM is the input beginning permutation (a permutation
% of the first $n$ integers).
% PROXTM is the two-mode $nrow \times ncol$ input proximity matrix.
% TARG is the $n \times n$ input target matrix.
% OUTPERM is the final permutation of SQUAREPROX with the
% cross-product index RAWINDEX
% with respect to TARG. ALLPERMS is a cell array containing INDEX
% entries corresponding to all the
% permutations identified in the optimization from ALLPERMS{1}
% = INPERM to ALLPERMS{INDEX} = OUTPERM.

pairwiseqa.m

function [outperm, rawindex, allperms, index] = ...
   pairwiseqa(prox, targ, inperm)

% PAIRWISEQA carries out an iterative
% Quadratic Assignment maximization task using the
% pairwise interchanges of objects in the
% permutation defining the row and column
% order of the data matrix.
% INPERM is the input beginning permutation
(a permutation of the first $n$ integers).
% PROX is the $n \times n$ input proximity matrix.
% TARG is the $n \times n$ input target matrix.
% OUTPERM is the final permutation of PROX with the cross-product index RAWINDEX
% with respect to TARG.
% ALLPERMS is a cell array containing INDEX entries corresponding to all the permutations identified in the optimization from ALLPERMS{1} = INPERM to ALLPERMS{INDEX} = OUTPERM.

**proxmon.m**

function [monproxpermut, vaf, diff] = proxmon(proxpermut, fitted)

% PROXMON produces a monotonically transformed proximity matrix (MONPROXPERMUT) from the order constraints obtained from each pair of entries in the input proximity matrix PROXPERMUT (symmetric with a zero main diagonal and a dissimilarity interpretation). MONPROXPERMUT is close to the $n \times n$ matrix FITTED in the least-squares sense; the variance accounted for (VAF) is how much variance in MONPROXPERMUT can be accounted for by FITTED; DIFF is the value of the least-squares criterion.

**proxmontm.m**

function [monproxpermuttm, vaf, diff] = ...
proxmontm(proxpermuttm, fittedtm)

% PROXMONTM produces a monotonically transformed two-mode proximity matrix (MONPROXPERMUTTM) from the order constraints obtained from each pair of entries in the input two-mode proximity matrix PROXPERMUTTM (with a dissimilarity interpretation). MONPROXPERMUTTM is close to the $nrow \times ncol$ matrix FITTEDTM in the least-squares sense; the variance accounted for (VAF) is how much variance in MONPROXPERMUTTM can be accounted for by FITTEDTM; DIFF is the value of the least-squares criterion.
Appendix A. Header comments for the mentioned m-files

**proxrand.m**

```matlab
function [randprox] = proxrand(prox)

% PROXRAND produces a symmetric proximity matrix
% with a zero main diagonal having
% entries that are a random permutation of those in the
% symmetric input proximity
% matrix PROX.
```

**proxrandtm.m**

```matlab
function [randproxtm] = proxrandtm(proxtm)

% PROXRANDTM produces a two-mode proximity matrix having
% entries that are a random permutation of
% those in the two-mode input proximity
% matrix PROXTM.
```

**proxstd.m**

```matlab
function [stanprox, stanproxmult] = proxstd(prox,mean)

% PROXSTD produces a standardized proximity matrix (STANPROX)
% from the input $n \times n$ proximity matrix
% (PROX) with zero main diagonal and a dissimilarity
% interpretation.
% STANPROX entries have unit variance (standard deviation of one)
% with a mean of MEAN given as an input number;
% STANPROXMULT (upper-triangular) entries have a sum of
% squares equal to $n(n-1)/2$.
```

**proxstdtm.m**

```matlab
function [stanproxtm, stanproxmulttm] = proxstdtm(proxtm,mean)

% PROXSTDTM produces a standardized two-mode
% proximity matrix (STANPROXTM) from the input
% $nrow \times ncol$ two-mode proximity matrix (PROXTM) with a
% dissimilarity interpretation.
% STANPROXTM entries have unit variance (standard deviation
% of one) with a mean of MEAN given as an input number;
% STANPROXMULTTM entries have a sum of squares equal to
% $nrow \times rcol$.

**randprox.m**

function [prox] = randprox(n)

% RANDPROX produces a random symmetric proximity matrix of size
% $n \times n$, with a zero main diagonal and entries uniform
% between 0 and 1.

**rotateqa.m**

function [outperm, rawindex, allperms, index] = ...
    rotateqa (prox, targ, inperm, kblock)

% ROTATEQA carries out an iterative
% Quadratic Assignment maximization task using the
% rotation of from 2 to KBLOCK (which is less than or
% equal to $n-1$) consecutive objects in
% the permutation defining the row and column order of the data
% matrix.
% INPERM is the input beginning permutation
% (a permutation of the first $n$ integers).
% PROX is the $n \times n$ input proximity matrix.
% TARG is the $n \times n$ input target matrix.
% OUTPERM is the final permutation of PROX with the cross-product
% index RAWINDEX with respect to TARG.
% ALLPERMS is a cell array containing INDEX entries corresponding
% to all the permutations identified in the optimization from
% ALLPERMS{1} = INPERM to ALLPERMS{INDEX} = OUTPERM.

**sarobfit.m**

function [fit, vaf] = sarobfit(prox, inperm)

% SAROBFIT fits a strongly anti-Robinson matrix using iterative
% projection to a symmetric proximity matrix in the $L_{2}$-norm.
% PROX is the input proximity matrix ($n \times n$ with a zero
% main diagonal and a dissimilarity interpretation);
% INPERM is a given permutation of the first $n$ integers;
% FIT is the least-squares optimal matrix (with
% variance-accounted-for of VAF) to PROX having a strongly
% anti-Robinson form for the row and column
% object ordering given by INPERM.

**sarobfnd.m**

function [find, vaf, outperm] = sarobfnd(prox, inperm, kblock)

% SAROBFND finds and fits a strongly
% anti-Robinson matrix using iterative projection to
% a symmetric proximity matrix in the $L_{2}$-norm based on a
% permutation identified through the use of iterative
% quadratic assignment.
% PROX is the input proximity matrix ($n \times n$ with a zero
% main diagonal and a dissimilarity interpretation);
% INPERM is a given starting permutation of the first $n$ integers;
% FIT is the least-squares optimal matrix (with
% variance-accounted-for of VAF) to PROX having a strongly
% anti-Robinson form for the row and column
% object ordering given by the ending permutation OUTPERM. KBLOCK
% defines the block size in the use the iterative
% quadratic assignment routine.

**targcir.m**

function [targcircular] = targcir(n)

% TARGCIR produces a symmetric proximity matrix of size
% $n \times n$, containing distances
% between equally and unit-spaced positions
% along a circle: targcircular(i,j) = min(abs(i-j),n-abs(i-j)).

**targfit.m**

function [fit, vaf] = targfit(prox,targ)

% TARGFIT fits through iterative projection a given set of equality
% and inequality constraints (as represented by the equalities and
% inequalities present among the entries in a target matrix
% TARG) to a symmetric proximity matrix in the $L_{2}$-norm.
% PROX is the input proximity matrix (with a zero main diagonal
% and a dissimilarity interpretation);
% TARG is a matrix of the same size as PROX;
% FIT is the least-squares optimal matrix (with
% variance-accounted-for of VAF) to PROX satisfying the equality and
% inequality constraints implicit in TARG.

**targlin.m**

function [targlinear] = targlin(n)

% TARGLIN produces a symmetric proximity matrix of size
% $n \times n$, containing distances
% between equally and unit-spaced positions
% along a line: targlinear(i,j) = abs(i-j).

**trimonscalqa.m**

function [outpermone, outpermtwo, outpermthree, coordone, ...
coordtwo, coordthree, fitone, fittwo, fitthree, addconone, ...
addcontwo, addconthree, vaf, monprox] = trimonscalqa(prox, targone,...
targtwo, targthree, inpermone, inpermtwo, inpermthree, kblock, nopt)

% TRIMONSCALQA carries out a tridimensional scaling of a symmetric
% proximity matrix using iterative quadratic assignment,
% plus it provides an optimal monotonic transformation
% (MONPROX) of the original input proximity matrix.
% PROX is the input proximity matrix (with a zero main diagonal
% and a dissimilarity interpretation);
% TARGONE is the input target matrix for the first dimension
% (usually with a zero main diagonal and with a dissimilarity
% interpretation representing equally-spaced locations
% along a continuum); TARGTWO is the input target
% matrix for the second dimension; TARGTHREE is the input target
% matrix for the third dimension;
% INPERMONE is the input beginning permutation for the
% first dimension (a permutation of the first $n$ integers);
% INPERMTWO is the input beginning
% permutation for the second dimension; INPERMTHREE is the input
% beginning permutation for the third dimension;
% the insertion and rotation routines use from 1 to KBLOCK
% (which is less than or equal to $n-1$) consecutive objects in
% the permutation defining the row and column orders of the data
% matrix; NOPT controls the confirmatory or exploratory fitting
% of the unidimensional scales; a value of NOPT = 0 will fit in
% a confirmatory manner the two scales.
% indicated by INPERMONE and INPERMTWO; a value of NOPT = 1
% uses iterative QA to locate the better permutations to fit;
% OUTPERMONE is the final object permutation for the first
% dimension; OUTPERMTWO is the final object permutation
% for the second dimension; OUTPERMTHREE is the final object
% permutation for the third dimension;
% COORDONE is the set of first dimension coordinates
% in ascending order; COORDTWO is the set of second dimension
% coordinates in ascending order; COORDTHREE is the set of
% second dimension coordinates in ascending order;
% ADDCONONE is the additive constant for the first
% dimensional model; ADDCONTWO is the additive constant
% for the second dimensional model; ADDCONTHREE is the additive
% constant for the third dimensional model;
% VAF is the variance-accounted-for in MONPROX by
% the tridimensional scaling.

triscalqa.m

function [outpermone,outpermtwo,outpermthree,coordone,...
coordtwo,coordthree,fitone,fittwo,fitthree,addconone,...
addcontwo,addconthree,vaf] = triscalqa(prox,targone,targtwo,...
targthree,inpermone,inpermtwo,inpermthree,kblock,nopt)

% TRISCALQA carries out a tridimensional scaling of a symmetric
% proximity matrix using iterative quadratic assignment.
% PROX is the input proximity matrix (with a zero main diagonal
% and a dissimilarity interpretation);
% TARGONE is the input target matrix for the first dimension
% (usually with a zero main diagonal and with a dissimilarity
% interpretation representing equally-spaced locations along
% a continuum); TARGTWO is the input target
% matrix for the second dimension; TARGTHREE is the input
% target matrix for the third dimension;
% INPERMONE is the input beginning permutation for the first
% dimension (a permutation of the first $n$ integers);
% INPERMTWO is the input beginning permutation for the
% second dimension; INPERMTHREE is the input beginning
% permutation for the third dimension;
% the insertion and rotation routines use from 1 to KBLOCK
% (which is less than or equal to $n-1$) consecutive objects in
% the permutation defining the row and column orders of the data
% matrix; NOPT controls the confirmatory or exploratory fitting
% of the unidimensional scales; a value of NOPT = 0 will fit in
% a confirmatory manner the three scales
% indicated by INPERMONE and INPERMTWO;
% a value of NOPT = 1 uses iterative QA
% to locate the better permutations to fit.
% OUTPERMONE is the final object permutation for
% the first dimension; OUTPERMTWO is the final object permutation
% for the second dimension; OUTPERMTHREE is the final object
% permutation for the third dimension; COORDONE is the set of
% first dimension coordinates in ascending order;
% COORDTWO is the set of second dimension coordinates in ascending
% order; COORDTHREE is the set of third dimension coordinates
% in ascending order; ADDCONONE is the additive constant for the
% first dimensional model; ADDCONTWO is the additive
% constant for the second dimensional model; ADDCONTHREE is the additive
% VAF is the variance-accounted-for in PROX by the
% bidimensional scaling.

**ultracomptm.m**

function [ultracomp] = ultracomptm(ultraproxtm)

% ULTRACOMPTM provides a completion of a given two-mode ultrametric
% matrix to a symmetric proximity matrix satisfying the
% usual ultrametric constraints.
% ULTRAPROXTM is the $nrow \times ncol$ two-mode ultrametric matrix;
% ULTRACOMP is the completed symmetric
% $n \times n$ proximity matrix having the usual
% ultrametric pattern, for $n = nrow + ncol$.

**ultrafit.m**

function [fit,vaf] = ultrafit(prox,targ)

% ULTRAFIT fits a given ultrametric using iterative projection to
% a symmetric proximity matrix in the $L_{2}$-norm.
% PROX is the input proximity matrix (with a zero main diagonal
% and a dissimilarity interpretation);
% TARG is an ultrametric matrix of the same size as PROX;
% FIT is the least-squares optimal matrix (with
% variance-accounted-for of VAF) to PROX satisfying the ultrametric
% constraints implicit in TARG.
ultrafittm.m

function [fit,vaf] = ultrafittm(proxtm,targ)

% ULTRAFITTM fits a given (two-mode) ultrametric using iterative
% projection to a two-mode (rectangular) proximity matrix in the
% $L_{2}$-$\|\cdot\|$-norm.
% PROXTM is the input proximity matrix (with a dissimilarity
% interpretation); TARG is an ultrametric matrix of the same size
% as PROXTM; FIT is the least-squares optimal matrix (with
% variance-accounted-for of VAF) to PROXTM satisfying the
% ultrametric constraints implicit in TARG.

ultrafnd.m

function [find,vaf] = ultrafnd(prox,inperm)

% ULTRAFND finds and fits an ultrametric using iterative projection
% heuristically on a symmetric proximity matrix in the $L_{2}$-$\|\cdot\|$-norm.
% PROX is the input proximity matrix (with a zero main diagonal
% and a dissimilarity interpretation);
% INPERM is a permutation that determines the order in which the
% inequality constraints are considered;
% FIND is the found least-squares matrix (with variance-accounted-for
% of VAF) to PROX satisfying the ultrametric constraints.

ultrafndtm.m

function [find,vaf] = ultrafndtm(proxtm,inpermrow,inpermcol)

% ULTRAFNDTM finds and fits a two-mode ultrametric using
% iterative projection heuristically on a rectangular proximity
% matrix in the $L_{2}$-$\|\cdot\|$-norm.
% PROXTM is the input proximity matrix (with a
% dissimilarity interpretation);
% INPERMROW and INPERMCOL are permutations for the row and column
% objects that determine the order in which the
% inequality constraints are considered;
% FIND is the found least-squares matrix (with variance-accounted-for
% of VAF) to PROXTM satisfying the ultrametric constraints.
ultraorder.m

function [orderprox, orderperm] = ultraorder(prox)

% ULTRAORDER finds for the input proximity matrix PROX
% (assumed to be ultrametric with a zero main diagonal),
% a permutation ORDERPERM that displays the anti-
% Robinson form in the reordered proximity matrix
% ORDERPROX; thus, prox(orderperm, orderperm) = orderprox.

ultraplot.m

function [] = ultraplot(ultra)

% ULTRAPLOT gives a dendrogram plot for the input ultrametric
% dissimilarity matrix ULTRA.

unicirac.m

function [fit, vaf, outperm, addcon] = unicirac(prox, inperm, kblock)

% UNICIRAC finds and fits a circular
% unidimensional scale using iterative projection to
% a symmetric proximity matrix in the $L_2$-norm based on a
% permutation identified through the use of iterative
% quadratic assignment.
% PROX is the input proximity matrix ($n \times n$ with a
% zero main diagonal and a dissimilarity interpretation);
% INPERM is a given starting permutation (assumed to be around the
% circle) of the first $n$ integers;
% FIT is the least-squares optimal matrix (with
% variance-accounted-for of VAF) to PROX having a circular
% anti-Robinson form for the row and column
% object ordering given by the ending permutation OUTPERM.
% The spacings among the objects are given by the diagonal entries
% in FIT (and the extreme $(1,n)$ entry in FIT). KBLOCK
% defines the block size in the use of the iterative quadratic
% assignment routine.
% The additive constant for the model is given by ADDCON.

uniscalqa.m

function [outperm, rawindex, allperms, index, coord, diff] = ...
uniscalqa(prox, targ, inperm, kblock)

% UNISCALQA carries out a unidimensional scaling of a symmetric
% proximity matrix using iterative quadratic assignment.
% PROX is the input proximity matrix (with a zero main diagonal
% and a dissimilarity interpretation);
% TARG is the input target matrix (usually with a zero main
% diagonal and a dissimilarity interpretation representing
% equally-spaced locations along a continuum);
% INPERM is the input beginning permutation (a permutation of the
% first $n$ integers). OUTPERM is the final permutation of PROX
% with the cross-product index RAWINDEX
% with respect to TARG redefined as
$ = \{\text{abs(coord}(i) - \text{coord}(j))\}\}

% ALLPERMS is a cell array containing INDEX entries corresponding
% to all the permutations identified in the optimization from
% ALLPERMS{1} = INPERM to ALLPERMS{INDEX} = OUTPERM.
% The insertion and rotation routines use from 1 to KBLOCK
% (which is less than or equal to $n-1$) consecutive objects in
% the permutation defining the row and column order of the data
% matrix. COORD is the set of coordinates of the unidimensional
% scaling in ascending order;
% DIFF is the value of the least-squares loss function for the
% coordinates and object permutation.

uniscaltmac.m

function [fit, vaf, outperm, rowperm, colperm, addcon, coord] = ...
uniscaltmac(proxtm, inperm, kblock)

% UNISCALTMAC finds and fits a linear
% unidimensional scale using iterative projection to
% a two-mode proximity matrix in the $L_{2}$-norm based on a
% permutation identified through the use of iterative
% quadratic assignment.
% PROXTM is the input two-mode proximity matrix
% ($n_{a} \times n_{b}$ with a zero main diagonal
% and a dissimilarity interpretation);
% INPERM is a given starting permutation of the
% first $n = n_{a} + n_{b}$ integers;
% FIT is the least-squares optimal matrix (with
% variance-accounted-for of VAF) to PROXTM having a linear
% unidimensional form for the row and column
% object ordering given by the ending permutation OUTPERM.
% The spacings among the objects are given by the entries in FIT.
% KBLOCK defines the block size in the use of the iterative
% quadratic assignment routine.
% The additive constant for the model is given by ADDCON.
% ROWPERM and COLPERM are the resulting row and column
% permutations for the objects. The nrow + ncol coordinates
% (ordered with the smallest set at a value of zero)
% are given in COORD.
Appendix A. Header comments for the mentioned m-files

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