Psychology 594
Multivariate Analysis

Solve all the problems first by hand; redo the numerical analyses with MATLAB to check your results. Note: you don’t need to do any symbolic work in MATLAB; only reproduce the numerical results.

Homework I

Problem 1:
Let \( x' = [6, 2, 1] \) and \( y' = [-1, 3, 1] \).

(a) Graph the two vectors.

(b) Find (i) the length of \( x \), (ii) the angle between \( x \) and \( y \), and (iii) the projection of \( y \) on \( x \).

(c) Because \( \bar{x} = 3 \) and \( \bar{y} = 1 \), graph the mean centered vectors, \( [6 - 3, 2 - 3, 1 - 3] = [3, -1, -2] \) and \( [-1 - 1, 3 - 1, 1 - 1] = [-2, 2, 0] \). Calculate the correlation between the three observation pairs. Find the cosine of the angle between the two mean-corrected vectors, and comment on the relation to the correlation.

Problem 2:

Given the matrices
\[
A = \begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}
\]
perform the indicated multiplications:

(a) $5A$
(b) $AB$
(c) $B'A'$
(d) $C'A$
(e) Is $BA$ defined? If so, calculate it.

Problem 3:
Verify the following properties of the transpose and inverse when

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

(a) $(A')' = A$
(b) $(B')^{-1} = (B^{-1})'$
(c) $(AB)' = B'A'$
(d) $(AB)^{-1} = B^{-1}A^{-1}$

Problem 4:
Verify that

$$Q = \frac{1}{3} \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$$

is an orthogonal matrix; calculate $QQ'$, $Q'Q$, and $Q^{-1}$. 

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Homework II

Problem 1:
Let
\[
A = \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}
\]
(a) Is \( A \) symmetric?
(b) Is \( A \) positive definite?

Problem 2:
Let
\[
A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}
\]

a) Determine the eigenvalues and associated eigenvectors of \( A \). Find the spectral decomposition of \( A \).
b) Find \( A^{-1} \).

b) Compute the eigenvalues and eigenvectors of \( A^{-1} \), and write out the spectral decomposition of \( A^{-1} \). Compare this spectral decomposition with that for \( A \).

Problem 3:
A quadratic form \( \mathbf{x}'A\mathbf{x} \) is said to be positive definite if the matrix \( A \) is positive definite. Is the quadratic form, \( 4x_1^2 + 4x_2^2 - 6x_1x_2 \), positive definite?

Problem 4:
Determine the square root matrix $A^{1/2}$ using the matrix $A$ from the previous problem 3. Also, determine $A^{-1/2}$, and show that $A^{1/2}A^{-1/2} = A^{-1/2}A^{1/2} = I$.

Problem 5:
(a) Consider an arbitrary $n \times p$ matrix $A$. Then $A'A$ is a symmetric $p \times p$ matrix. Show that $A'A$ is necessarily positive semi-definite. (Hint: set $y = Ax$ so that $y'y = x'A'Ax$)

(b) Using the matrix

$$A = \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix}$$

(1) Calculate $AA'$ and obtain its eigenvalues and eigenvectors.

(2) Calculate $A'A$ and obtain its eigenvalues and eigenvectors. Check that the nonzero eigenvalues are the same as those in (1).

(3) Obtain the singular-value decomposition of $A$. 

Homework III

Problem 1:
Let \( X \) have covariance matrix

\[
\Sigma = \begin{bmatrix}
25 & 0 & 2 \\
0 & 1 & 1 \\
2 & 1 & 9
\end{bmatrix}
\]

(a) Determine \( \rho \) (the correlation matrix) and \( V^{1/2} \) (a diagonal matrix containing the standard deviations along the main diagonal).

(b) Multiply your matrices to check the relation \( V^{1/2} \rho V^{1/2} = \Sigma \).

(c) Find \( \rho_{13} \).

(d) Find the correlation between \( X_1 \) and \( \frac{1}{4}X_2 + \frac{1}{2}X_3 \).

Problem 2:
(a) Derive expressions for the mean and variances of the following linear combinations in terms of the means and covariances of the random variable \( X_1, X_2, \) and \( X_3 \).

1) \( X_1 - 2X_2 \)
2) \( -X_1 + 3X_2 \)
3) \( X_1 + X_2 + X_3 \)
4) \( X_1 + 2X_2 - X_3 \)
5) \( 3X_1 - 4X_2 \), when \( X_1 \) and \( X_2 \) are independent random variables.

(b) The random vector \( \mathbf{X}' = [X_1, X_2, X_3, X_4] \) has mean vector \([4, 3, 2, 1]\) and variance-covariance matrix
Let $X^{(1)} = [X_1, X_2]'$ and $X^{(2)} = [X_3, X_4]'$, and let $A = \begin{bmatrix} 1 & 2 \\ 9 & -2 \\ -2 & 6 \\ -2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

Find: $E(X^{(1)}); E(AX^{(1)}); E(X^{(2)}); E(BX^{(2)}); Cov(X^{(1)}); Cov(AX^{(1)}); Cov(X^{(2)}); Cov(BX^{(2)}); Cov(X^{(1)}, X^{(2)}); Cov(AX^{(1)}, BX^{(2)}).$

For

$A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$

find the maximum value of $x'Ax$ for $x'x = 1$.

Problem 3:
Give your own numerically specified $4 \times 3$ matrix, say, $A$, and do the MATLAB operations of rank, corrcoef, cov, mean, median, std, and sum on $A$, and inv, trace, det, eig, svd, poly, and sqrtm on $A'A$. 

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