Greenhouse, S. W., & Geisser, S. (1959). On methods in the analysis of profile data. *Psychometrika*, *24*, 95–112 (3622 citations in Google Scholar as of 4/1/2016).

To obtain a good sense of the importance of this highly cited Greenhouse and Geisser paper from the 1950s, it is well to remember the state of statistical computation at that time. It was done mostly with mechanical calculators and not by electronic computers with stored programs as is common today. Thus, it was a major contribution to the conduct of applied statistics when some particularly difficult analysis task could be reframed so that it might be done computationally with existing mechanical calculators.[1]

Profile data as discussed in the Greenhouse and Geisser paper can be characterized by subjects within groups observed over a battery of tests (or one test, say, that is repeatedly measured over a number of occasions). Three questions are typically of interest:

a) are the group profiles "parallel"? — here, "parallel" refers to the group means being equidistant at each measurement occasion;

b) assuming that profiles are parallel, are they also at the same level (that is, are they "coincident")?

c) assuming coincident profiles, are they also "horizontal"?

Adopting some of the notation of Greenhouse and Geisser, it is assumed that p tests, $x_1, \ldots, x_p$, are given to each of n_k individuals ($k = 1, 2, \ldots, g$) in each of g established groups. An individual i in group k is said to have the profile, $(x_{i1k}, \ldots, x_{ijk}, \ldots, x_{ipk})$, for $1 \leq i \leq n_k$, $1 \leq j \leq p$, and $1 \leq k \leq g$. The group k profile is represented by $(\bar{x}_{\cdot 1k}, \bar{x}_{\cdot 2k}, \ldots, \bar{x}_{\cdot pk})$. It is assumed that each individual profile is a random vector sampled from a p-variate normal distribution with arbitrary variance-covariance matrix. It is also assumed implicitly that the p (test) variables are commensurable (that is, the variables have the same metric). This last assumption allows meaning to be given to the question of whether the profiles have the same "shape" (here, profiles are said to have the same "shape" when they are parallel and the tests are

[1] For another *Psychometrika* paper on approximate methods for the analysis of repeated measures (but written some twenty years after Greenhouse and Geisser), see the informative review by Huynh Huynh, "Some approximate tests for repeated measurement designs," *Psychometrika* (1978, *43*, 161–175).

commensurable).

The question of profiles having the same shape may be best explained using just two group profiles, $(\bar{x}_{\cdot 11}, \bar{x}_{\cdot 21}, \ldots, \bar{x}_{\cdot p1})$ and $(\bar{x}_{\cdot 12}, \bar{x}_{\cdot 22}, \ldots, \bar{x}_{\cdot p2})$. Letting $d_1, d_2, \ldots, d_p$ represent the differences between groups means for each of the tests, a necessary and sufficient condition for the two profiles to be parallel is for $d_1 = d_2 = \cdots = d_p$. But this equality of the differences is exactly what is meant by a lack of interaction between groups and tests. In other words, an evaluation of the group-test interaction is really an evaluation of whether group profiles have the same shape.

By assuming the particular form for the variance-covariance matrix called compound symmetry (where the tests have equal variances and are equally correlated in pairs), the classical mixed model for g samples is generated. The resulting analysis-of-variance table would appear as follows:

Source	df	Sum Squares	Mean Squares
Tests	$p-1$	SSTests	MSTests
Groups	$g-1$	SSGroups	MSGroups
Individuals (w. groups)	$N-g$	SSInd (w. groups)	MSInd (w. groups)
Group x Test	$(p-1)(g-1)$	SS(GxT)	MS(GxT)
Ind x Test (w. groups)	$(p-1)(N-g)$	SS(IxT) (w. groups)	MS(IxT) (w. groups)

Based on this analysis-of-variance table, there are three F-tests for the three conjectures of interest:

a) Parallel (same shape) group profiles:

MS(GxT)/MS(IxT) (w. groups) $\sim \mathrm{F}_{(p-1)(g-1),(p-1)(N-g)}$

Here, the total number of observations, $N = (\sum_{k=1}^{g} n_k)$.

b) Coincident group profiles:

MSGroups/MSInd (w. groups) $\sim$ $F_{(g-1),(N-g)}$

This test is equivalent to first obtaining a single score for each individual by summing over tests and then performing a one-way analysis-of-variance on the resulting scores.

c) Horizontal group profiles:

MSTests/MS(IxT) (w. groups) $\sim$ $F_{(p-1),(p-1)(N-g)}$

When the assumption of compound symmetry for the covariance matrix does not hold, the F-tests for parallel group profiles and for horizontal group profiles are no longer valid. In fact, both tests are too liberal and would reject the corresponding (null) hypotheses too often; or, in other words, the obtained p-values are too small. The F-test for coincident group profiles is not affected and remains appropriate because it is based on summed scores for individuals. The genius of the Greenhouse-Geisser paper is that it showed how to correct the degrees-of-freedom for the tests for parallel and horizontal profiles based on an estimated function, ϵ, obtained from the (sample) variances and covariances among the tests (which is no longer assumed to have a compound symmetry form). For the parallel profiles test, the F-distribution used would be $F_{\epsilon(p-1)(g-1),\ \epsilon(p-1)(N-g)}$; for the horizontal group profiles test, the F-distribution used would be $F_{\epsilon(p-1),\ \epsilon(p-1)(N-g)}$. Because $1 \geq \epsilon$, the numerator and denominator degrees-of-freedom are both discounted, making the tests more conservative (and, therefore, more appropriate when compound symmetry does not hold); when compound symmetry does hold, $\epsilon = 1$, and no discounting of the degrees-of-freedom occurs.

The Greenhouse-Geisser paper also includes a lower bound for ϵ (that is, $\epsilon > \frac{1}{p-1}$). Thus, it is possible to discount the degrees-of-freedom maximally, giving the most conservative tests we would need: use $F_{g-1,N-g}$ to test for parallel profiles; use $F_{1,N-g}$ to test for horizontal profiles. The argument then continues as follows: if one rejects with the maximally discounted degrees-of-freedom, rejection would also occur with any value of ϵ greater than its lower bound.[2]

[2]It should be noted that the Greenhouse-Geisser *Psychometrika* paper does not include proofs for the

As hinted at in the beginning of this note, the Greenhouse-Geisser paper was important for the analysis of profile data in an era when computation was rather primitive and done without electronic computers. The mixed model for multiple groups assuming compound symmetry leads to an analysis-of-variance table that was well within the numerical capabilities of the mechanical calculators of the time. More importantly, the various tests could be modified to mitigate the effects of having to make the compound symmetry assumption. As is now well-known to anyone taking an applied multivariate analysis class (based, for example, on a text like Johnson and Wichern, *Applied Multivariate Statistical Analysis*), profile analysis can be done in an alternative manner with various multivariate analysis-of-variance techniques on difference scores. This approach requires significant numerical effort involving sample variance-covariance matrices that soon becomes prohibitive without computers and currently available statistical routines (in SYSTAT, SPSS, R, and Matlab, for example).

There is still one very important use for the Greenhouse-Geisser approach even in the face of all the computational power now commonly available. This is where the number of individuals is less than the number of tests; in these instances, the multivariate procedures are impossible to carry out because the necessary degrees-of-freedom are zero or negative for some of the F-approximations. Examples of profile analysis data where subjects are fewer than the number of observation occasions, abound in neuroimaging analyses done using fMRI data.

In closing we give two comments on the Greenhouse-Geisser paper from the authors themselves. The first is from Greenhouse on the paper being named a Science and Social Science Citation Classic (July, 1982):

> Seymour Geisser joins me in expressing our pleasure in learning that our 1959 paper is now a Citation Classic. Each of us was aware that our procedures were being applied because of the many letters and calls we received over the years. In preparing for this statement, I reread the paper. I must say it reads very well and is quite lucid in its exposition. I believe we have the then-editors of *Psychometrika* to thank for this in that, contrary to editorial

explicit form that ϵ should take nor for its lower bound. Formal demonstrations are given in a companion 1958 paper in *The Annals of Mathematical Statistics*, with the authors listed in the traditional alphabetical order: Seymour Geisser and Samuel W. Greenhouse, "An extension of Box's result on the use of the F distribution in multivariate analysis" (*29*, 885–891). As is common for an *Annals* paper, the proofs are extremely cryptic and given with little or no interpolated verbal explanation or discussion.

strictures currently imposed, they did not demand that we shorten the paper. The pace of the exposition and the examples presented made it possible for any interested reader to apply the methods we were describing.

It is interesting to note that Paul Horst and Dorothy Adkins were the two co-editors of *Psychometrika* when the Greenhouse-Geisser paper was reviewed and published.

The second not-so-nice comment is by Geisser taken from an interview published in *Statistical Science* in 2007 (*22*, pp. 627–628):

Wes: What kind of problems did you work on at NIH?

Seymour: Well, at NIH, with Sam Greenhouse, I wrote my most infamous paper. "Infamous," I say because it wasn't a very important or very hard paper. It was just a paper that seemed to have caught on with social scientists and some medical people. It was just this profile analysis paper which ended up being a citation classic, which means that it had a lot of citations. It still has more citations than all of my [other] papers altogether. [Laughs all around.]

Wes: This was the Greenhouse-Geisser paper?

Seymour: There are two papers. The first paper was in the *Annals*, which actually worked out all of the quadratic forms, their expectations, and the mathematics. And the second was in *Psychometrika* and that was the citation classic. That was just to show the methodology—how to use this. It wasn't a very big deal. I worked much harder on other papers and I think I produced much better work. But de *gustibus non disputandum est* [there's no accounting for taste].