Two-way fixed-effects analysis of variance with replication:

In one-way ANOVA, we had one set of treatments and our concern was whether the means differed significantly from one another (using the omnibus F-test)

For example:

Group 1	Group 2	Group 3
Drug 1	Drug 2	Drug 3

We assigned subjects at random to these groups, gave them a certain drug, and measured some dependent variable, e.g., blood pressure, temperature, etc.

A one-way ANOVA tells us whether the average values for each of the groups are significantly different from one another.

We now wish to generalize to what is callled a two-way fixed effects analysis of variance

1) Two-way because we categorize on the basis of two factors rather than one, e.g.,

			factor 1	
		Drug 1	Drug 2	Drug 3
factor 2	Male			
	Female			

This is called a (2 by 3) Factorial design:

factor 2 (sex) is a status factor; factor 1 (drugs) is an experimental factor

We assign subjects at random within each sex

2) Fixed effects: this has the same meanings as before. We wish to generalize only to the levels of the factors that actually appear in the design

3) Replication: more than one subject in each group

3) If we have an equal number of observations in cells, the design is said to be "orthogonal"

A factorial design could be viewed as two separate one-way analyses:

between the three levels of drugs, collapsing over sex; or between the two levels of sex, collapsing over drugs

The advantage that the two-way design has is in allowing you to separate out effects that could appear only at one level of one factor and not the other.

As an example, suppose we have means in our design of the following form:

	Drug 1	Drug 2	Drug 3	
Male	100	80	90	90
Female	80	100	90	90
	90	90	90	

Suppose a high scores means a bad effect, i.e., 100 means death; 90 means sickness; 80 means health

What drug would you give to a male or a female?

But yet, the *main effect* means are all the same.

Another example:

	Method 1	Method 2	
Low SES	50	10	30
High SES	10	50	30
	30	30	

This is an example of an "aptitude-treatment" interaction: give method 1 to low SES students and method 2 to high SES students

For years, the search for powerful aptitude-treatment interactions was like the search for the holy grail – they were never found

Two factors are said to be "completely crossed" if all levels of one appear with each level of the other

Two factors are said to be "nested" if levels of one factor only occur within certain levels of another

School and method are “confounded” in the following design:

School A		School B	
Method 1	Method 2	Method 3	Method 4

In the following, School and Method are completely crossed (i.e., no nesting)

School A		School B	
Method 1	Method 2	Method 1	Method 2

We will worry about completely crossed (and unnested) factors for now

Linear model:

One-way: $Y_{ij} = \mu_i + \epsilon_{ij} = \mu. + \tau_i + \epsilon_{ij}$

where $\tau_i = \mu_i - \mu.$

Two-way: $Y_{ijk} = \mu_{ij} + \epsilon_{ijk}$,

for the i^{th} level on Factor A;

for the j^{th} level on Factor B;

for the k^{th} subject;

$1 \leq i \leq a$; $1 \leq j \leq b$; $1 \leq k \leq n$;

$\epsilon_{ijk} \sim N(0, \sigma^2)$, and all are independent

We assume the n's are equal so $n_T = abn$

Define:

the grand mean as: $\mu_{\cdot\cdot} = \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} \mu_{ij}$

the mean in row i: $\mu_{i\cdot} = \frac{1}{b} \sum_{j=1}^{b} \mu_{ij}$

the mean in column j: $\mu_{\cdot j} = \frac{1}{a} \sum_{i=1}^{a} \mu_{ij}$

the main effect of the i^{th} level of factor A: $\alpha_i = \mu_{i\cdot} - \mu_{\cdot\cdot}$

the main effect of the j^{th} level of factor B: $\beta_j = \mu_{\cdot j} - \mu_{\cdot\cdot}$

the interaction effect of the i^{th} level of A and the j^{th} level of B:

$(\alpha\beta)_{ij} = \mu_{ij} - \alpha_i - \beta_j - \mu_{\cdot\cdot} =$

$\mu_{ij} - \mu_{i\cdot} - \mu_{\cdot j} + \mu_{\cdot\cdot}$

Note: $\mu_{ij} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + (\alpha\beta)_{ij}$

Thus: $Y_{ijk} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$

If all the $(\alpha\beta)_{ij} = 0$, the model is said to be "additive"

We have restrictions on the various effects:

$\sum_{i=1}^{a} \alpha_i = 0$; $\sum_{j=1}^{b} \beta_j = 0$; $\sum_{i=1}^{a}(\alpha\beta)_{ij} = 0$

$\sum_{j=1}^{b}(\alpha\beta)_{ij} = 0$

To estimate the various effects just plug the sample mean values into their definitions

It is possible to graph the means and visually look for main effects and interaction

Suppose we place (mark) the levels of Factor A along a horizontal axis and plot the means for the levels of Factor B as lines in the two-dimensional plot (the vertical axis is calibrated according to the values of the means)

If the lines for the levels of Factor B are all horizontal, there is no main effect for A

If the lines for the levels of Factor B are all coincident (i.e., they lie on top of each other), there is no main effect for B

If the lines for the levels of Factor B are all parallel, there is no interaction

If the lines cross no matter which factor is placed along the horizontal axis, the interaction is said to be "disordinal"

Think Mary Poppins: horizont(inality); parallel(lility); coincident(aly)

To get the necessary sums of squares and the ANOVA table, notice the identity:

$$(Y_{ijk} - \bar{Y}_{\cdots}) = (Y_{ijk} - \bar{Y}_{ij\cdot}) + (\bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdots}) +$$

$$(\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdots}) + (\bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\cdots})$$

The original deviation from the grand mean is split into terms for error, the two main effects, and the interaction

SSTO (Sum of Squares Total):

$$\sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (Y_{ijk} - \bar{Y}_{\cdots})^2$$

SSE (Sum of Squares Error):

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(Y_{ijk} - \bar{Y}_{ij\cdot})^2$$

SSA (Sum of Squares for Factor A):

$$nb\sum_{i=1}^{a}(\bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot})^2$$

SSB (Sum of Squares for Factor B):

$$na\sum_{j=1}^{b}(\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot\cdot\cdot})^2$$

SSAB (Sum of Squares for Interaction):

$$n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\cdot\cdot\cdot})^2$$

If I view my factorial design as a one-way with ab levels, then the SSTR is the sum of SSA, SSB, and SSAB

Here's what the ANOVA table would look like:

Source	SS	df	MS	F
Factor A	SSA	$a-1$	MSA	$\frac{MSA}{MSE} \sim F_{a-1,ab(n-1)}$
Factor B	SSB	$b-1$	MSB	$\frac{MSB}{MSE} \sim F_{b-1,ab(n-1)}$
A $\times$ B	SSAB	$(a-1)(b-1)$	MSAB	$\frac{MSAB}{MSE} \sim F_{(a-1)(b-1),ab(n-1)}$
Error	SSE	$ab(n-1)$	MSE	
Total	SSTO	$abn-1$		

Computational Formulas:

Need the following:

(1) Total Sum: $\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} Y_{ijk}$

(2) Total Sum of Squares: $\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} Y_{ijk}^2$

(3) Sum Rows: $\sum_{j=1}^{b}\sum_{k=1}^{n} Y_{ijk}$

(4) Sum Columns: $\sum_{i=1}^{a}\sum_{k=1}^{n} Y_{ijk}$

(5) Sum Cell: $\sum_{k=1}^{n} Y_{ijk}$

$$\text{SSTO} = (2) - \frac{(1)^2}{nab}$$

$$\text{SSA} = \frac{\sum (3)^2}{bn} - \frac{(1)^2}{nab}$$

$$\text{SSB} = \frac{\sum (4)^2}{an} - \frac{(1)^2}{nab}$$

$$\text{SSE} = (2) - \frac{\sum (5)^2}{n}$$

The SSAB for interaction is obtained by subtraction

Regression Approach to two-way analysis of variance:

Because we use the "effect form" of the linear model, we have to express one α in terms of the other α's, one β in terms of the other β's, and $b + a - 1$ of the $\alpha\beta$'s in terms of the other $\alpha\beta$'s

As an example, consider the simple 2×2 design (the cell means are indicated):

		Factor B	
		1	2
Factor A	1	μ_{11}	μ_{12}
	2	μ_{21}	μ_{22}

$$\boldsymbol{Y} = \begin{bmatrix} 11 \\ 12 \\ 21 \\ 22 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} \mu. \\ \alpha_1 \\ \beta_1 \\ (\alpha\beta)_{11} \end{bmatrix} + \boldsymbol{\epsilon}$$

All of this can be generalized to an arbitrary Factorial Design ($A \times B$) with ab total cells

All of the various tests can now be done by comparing full and reduced models (even, by the way, when the cell sizes are not all the same)

Redoing two-way ANOVA in terms of orthogonal contrasts:

Consider the 2×2 design:

		Factor B	
		1	2
Factor A	1	μ_{11}	μ_{12}
	2	μ_{21}	μ_{22}

Lay this out in a one-way ANOVA with four cells:

	Group 1	Group 2	Group 3	Group 4
	μ_{11}	μ_{12}	μ_{21}	μ_{22}
L_1	1	1	-1	-1
L_2	1	-1	1	-1
L_3	1	-1	-1	1

The three contrasts are mutually orthogonal and $SS(L_1) = SSA$; $SS(L_2) = SSB$; $SS(L_3) = SSAB$

Also, $SSTR = SS(L_1) + SS(L_2) + SS(L_3) = SSA + SSB + SSAB$